A Handbook of Applied EM Derivation Tables

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Part I

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1- Time-Harmonic Electrodynamics

Table 1. Derivation of the Complete Time-Harmonic Solutions

		$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_e$	(1-a)
		$\boldsymbol{\nabla}\times\boldsymbol{E}=-\boldsymbol{M}-j\omega\boldsymbol{B}$	(1-b)
		$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\rho_m$	(1-c)
		$\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}+j\omega\boldsymbol{D}$	(1-d)
		$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})=0$	VI-1
		$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi)=\boldsymbol{0}$	VI-2
		$-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})=\boldsymbol{\nabla}^2\boldsymbol{A}$	VI-3
		Electric Sources	
		$\rho_m=0,\boldsymbol{M}=\boldsymbol{0},\boldsymbol{E}=\boldsymbol{E}_A,\boldsymbol{H}=\boldsymbol{H}_A$	(2)
(1), (2)	→	$\boldsymbol{\nabla}\cdot\boldsymbol{D}_A=\rho_e$	(3-a)
		$\boldsymbol{\nabla}\times\boldsymbol{E}_A=-j\omega\boldsymbol{B}_A$	(3-b)
		$\boldsymbol{\nabla}\cdot\boldsymbol{B}_A=0$	(3-c)
		$\boldsymbol{\nabla}\times\boldsymbol{H}_A=\boldsymbol{J}+j\omega\boldsymbol{D}_A$	(3-d)
VI-1, (3-c)	→	$\boldsymbol{H}_A=\frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A}$	(4)
(3-d), (4)	→	$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A}=\mu\boldsymbol{J}+j\omega\mu\varepsilon\boldsymbol{E}_A$	(5)
VI-3, (5)	→	$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\boldsymbol{\nabla}^2\boldsymbol{A}=\mu\boldsymbol{J}+j\omega\mu\varepsilon\boldsymbol{E}_A$	(6)
(3-b), (4)	→	$\boldsymbol{\nabla}\times(\boldsymbol{E}_A+j\omega\boldsymbol{A})=\boldsymbol{0}$	(7)
VI-2, (7)	→	$\boldsymbol{E}_A+j\omega\boldsymbol{A}\equiv-\boldsymbol{\nabla}\phi_e$	(8)
(8)	→	$\boldsymbol{E}_A=-j\omega\boldsymbol{A}-\boldsymbol{\nabla}\phi_e$	(9)
(6), (9)	→	$\boldsymbol{\nabla}^2\boldsymbol{A}+\omega^2\mu\varepsilon\boldsymbol{A}-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}+j\omega\mu\varepsilon\phi_e)=-\mu\boldsymbol{J}$	(10)
		Lorentz Gauge	
		$\boldsymbol{\nabla}\cdot\boldsymbol{A}\equiv-j\omega\mu\varepsilon\phi_e$	(11)
(9), (11)	→	$\boldsymbol{E}_A=-j\omega\boldsymbol{A}-j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$	(12)
(10), (11)	→	$\boldsymbol{\nabla}^2\boldsymbol{A}+\omega^2\mu\varepsilon\boldsymbol{A}=-\mu\boldsymbol{J}$	(13)
		Magnetic Sources	
		$\rho_e=0,\boldsymbol{J}=\boldsymbol{0},\boldsymbol{E}=\boldsymbol{E}_F,\boldsymbol{H}=\boldsymbol{H}_F$	(14)
(1), (14)	→	$\boldsymbol{\nabla}\cdot\boldsymbol{D}_F=0$	(15-a)
		$\boldsymbol{\nabla}\times\boldsymbol{E}_F=-\boldsymbol{M}-j\omega\boldsymbol{B}_F$	(15-b)
		$\boldsymbol{\nabla}\cdot\boldsymbol{B}_F=\rho_m$	(15-c)
		$\boldsymbol{\nabla}\times\boldsymbol{H}_F=j\omega\boldsymbol{D}_F$	(15-d)

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VI-1, (15-a) → $\boldsymbol{E}_F = -\frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{F}$ (16)

(15-b), (16) → $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{F} = \varepsilon\boldsymbol{M} + j\omega\mu\varepsilon\boldsymbol{H}_F$ (17)

VI-3, (17) → $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \boldsymbol{\nabla}^2\boldsymbol{F} = \varepsilon\boldsymbol{M} + j\omega\mu\varepsilon\boldsymbol{H}_F$ (18)

(15-d), (16) → $\boldsymbol{\nabla}\times(\boldsymbol{H}_F + j\omega\boldsymbol{F}) = \boldsymbol{0}$ (19)

VI-2, (19) → $\boldsymbol{H}_F + j\omega\boldsymbol{F} \equiv -\boldsymbol{\nabla}\phi_m$ (20)

(20) → $\boldsymbol{H}_F = -j\omega\boldsymbol{F} - \boldsymbol{\nabla}\phi_m$ (21)

(18), (21) → $\boldsymbol{\nabla}^2\boldsymbol{F} + \omega^2\mu\varepsilon\boldsymbol{F} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F} + j\omega\mu\varepsilon\phi_m) = -\varepsilon\boldsymbol{M}$ (22)

Lorentz Gauge

$\boldsymbol{\nabla}\cdot\boldsymbol{F} \equiv -j\omega\mu\varepsilon\phi_m$ (23)

(21), (23) → $\boldsymbol{H}_F = -j\omega\boldsymbol{F} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F})$ (24)

(22), (23) → $\boldsymbol{\nabla}^2\boldsymbol{F} + \omega^2\mu\varepsilon\boldsymbol{F} = -\varepsilon\boldsymbol{M}$ (25)

Complete Time-Harmonic Solutions

(12), (16) →
$$\boldsymbol{E} = \boldsymbol{E}_A + \boldsymbol{E}_F$$
$$= -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{F}$$

(4), (24) →
$$\boldsymbol{H} = \boldsymbol{H}_A + \boldsymbol{H}_F$$
$$= \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A} - j\omega\boldsymbol{F} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F})$$

(13) → $\boldsymbol{\nabla}^2\boldsymbol{A} + \omega^2\mu\varepsilon\boldsymbol{A} = -\mu\boldsymbol{J}$

(25) → $\boldsymbol{\nabla}^2\boldsymbol{F} + \omega^2\mu\varepsilon\boldsymbol{F} = -\varepsilon\boldsymbol{M}$

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Table 2. Derivation of the Power Balance Equation and the Poynting Vector

$$\nabla \cdot \boldsymbol{D} = \rho_e \tag{1-a}$$
$$\nabla \times \boldsymbol{E} = -\boldsymbol{M} - j\omega\mu\boldsymbol{H} \tag{1-b}$$
$$\nabla \cdot \boldsymbol{B} = \rho_m \tag{1-c}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E} \tag{1-d}$$

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{B} - (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{A} \qquad \text{VI}$$

Conduction (Ohmic) and Dielectric Losses

Conduction losses are taken into account by first splitting the electric current density $\boldsymbol{J}$ into impressed and conduction components, i.e., $\boldsymbol{J} = \boldsymbol{J}_{imp} + \boldsymbol{J}_{cnd}$ and then requiring that $\boldsymbol{J}_{cnd} = \sigma\boldsymbol{E}$ in which σ is the conductivity of the medium. Dielectric losses are taken into account by introducing the complex permittivity, i.e., $\varepsilon = \varepsilon' - j\varepsilon''$ in which $\varepsilon' \equiv \varepsilon_0\varepsilon_r'$ and $\varepsilon'' \equiv \varepsilon_0\varepsilon_r''$. (2)

Using the above modifications, equation (1-d) can be rewritten as $\nabla \times \boldsymbol{H} = \boldsymbol{J}_{imp} + j\omega\varepsilon_c\boldsymbol{E}$ in which $\varepsilon_c = \varepsilon'\left(1 - j\frac{\sigma_{eff}}{\omega\varepsilon'}\right)$ and $\sigma_{eff} \equiv \sigma + \omega\varepsilon''$.

Notation Simplification

In what follows, $\boldsymbol{J}_{imp}$ is simply denoted as $\boldsymbol{J}$. (3)

(1-d), (2), (3) $\rightarrow$ $$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \sigma_{eff}\boldsymbol{E} + j\omega\varepsilon'\boldsymbol{E} \tag{4}$$

The Power Balance Equation

(1-b) $\rightarrow$ $$\frac{1}{2}\boldsymbol{H}^* \cdot (\nabla \times \boldsymbol{E}) = -\frac{1}{2}\boldsymbol{H}^* \cdot \boldsymbol{M} - j\frac{1}{2}\omega\mu\boldsymbol{H} \cdot \boldsymbol{H}^* \tag{5-a}$$

(4) $\rightarrow$ $$\frac{1}{2}\boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}^*) = \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{J}^* + \frac{1}{2}\sigma_{eff}\boldsymbol{E} \cdot \boldsymbol{E}^* - j\frac{1}{2}\omega\varepsilon'\boldsymbol{E} \cdot \boldsymbol{E}^* \tag{5-b}$$

VI, (5) $\rightarrow$ $$\begin{aligned} \nabla \cdot \left(\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\right) &= -\frac{1}{2}(\boldsymbol{H}^* \cdot \boldsymbol{M} + \boldsymbol{E} \cdot \boldsymbol{J}^*) \\ &\quad -\frac{1}{2}\sigma_{eff}\boldsymbol{E} \cdot \boldsymbol{E}^* \\ &\quad -j2\omega\left(\frac{1}{4}\mu\boldsymbol{H} \cdot \boldsymbol{H}^* - \frac{1}{4}\varepsilon'\boldsymbol{E} \cdot \boldsymbol{E}^*\right) \end{aligned} \tag{6}$$

(6) $\rightarrow$ $$\begin{aligned} -\frac{1}{2}(\boldsymbol{H}^* \cdot \boldsymbol{M} + \boldsymbol{E} \cdot \boldsymbol{J}^*) &= \nabla \cdot \left(\frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*\right) \\ &\quad + \frac{1}{2}\sigma_{eff}\boldsymbol{E} \cdot \boldsymbol{E}^* \\ &\quad + j2\omega\left(\frac{1}{4}\mu\boldsymbol{H} \cdot \boldsymbol{H}^* - \frac{1}{4}\varepsilon'\boldsymbol{E} \cdot \boldsymbol{E}^*\right) \end{aligned}$$

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The Poynting Vector

The Instantaneous Poynting Vector

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{r},t) &\equiv \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) \\ &= \frac{1}{2}(\boldsymbol{E}(\boldsymbol{r})e^{j\omega t}+\boldsymbol{E}^{*}(\boldsymbol{r})e^{-j\omega t})\times\frac{1}{2}(\boldsymbol{H}(\boldsymbol{r})e^{j\omega t}+\boldsymbol{H}^{*}(\boldsymbol{r})e^{-j\omega t}) \\ &= \frac{1}{2}Re\left[\boldsymbol{E}(\boldsymbol{r})e^{j\omega t}\times\boldsymbol{H}^{*}(\boldsymbol{r})e^{-j\omega t}\right]+\frac{1}{2}Re\left[\boldsymbol{E}(\boldsymbol{r})e^{j\omega t}\times\boldsymbol{H}(\boldsymbol{r})e^{j\omega t}\right] \\ &= \frac{1}{2}Re[\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}^{*}(\boldsymbol{r})]+\frac{1}{2}Re\left[\left(\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})\right)e^{j2\omega t}\right] \end{aligned} \quad (7)$$

In Equation (7), the expression $Re\left[\frac{1}{2}\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}^{*}(\boldsymbol{r})\right]$ is referred to as the time-average Poynting vector (or simply the Poynting vector) and is denoted by $\boldsymbol{P}_{av}$.

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Table 3. Demonstration of the Temporal Variations of RHCP and LHCP Polarizations

Since the field must be observed as it is traveling away from the observer, the vertical axis is chosen as the x axis and the horizontal axis is chosen as the y axis. With this choice, the z axis points through the page and a Uniform Plane Wave that is propagating along the positive z axis is propagating away from the observer that is looking at the page.

Right-Hand Circular Polarized (RHCP)

$$\boldsymbol{E}_0 = E_0\left(j\boldsymbol{a}_x + \boldsymbol{a}_y\right) \tag{1}$$

(1) → $$\begin{aligned} \boldsymbol{E}_0(t) &= Re\left[E_0\left(j\boldsymbol{a}_x + \boldsymbol{a}_y\right)e^{j\omega t}\right] \\ &= E_0\left(-\sin\omega t\,\boldsymbol{a}_x + \cos\omega t\,\boldsymbol{a}_y\right) \end{aligned} \tag{2}$$

(2) →

t	=	0	ωt	=	0	$\boldsymbol{E}_0(t)$	= $E_0(\boldsymbol{a}_y)$
t	=	$\frac{T}{4}$	ωt	=	$\frac{\pi}{2}$	$\boldsymbol{E}_0(t)$	= $E_0(-\boldsymbol{a}_x)$
t	=	$\frac{T}{2}$	ωt	=	π	$\boldsymbol{E}_0(t)$	= $E_0(-\boldsymbol{a}_y)$
t	=	$\frac{3T}{4}$	ωt	=	$\frac{3\pi}{2}$	$\boldsymbol{E}_0(t)$	= $E_0(\boldsymbol{a}_x)$
t	=	T	ωt	=	2π	$\boldsymbol{E}_0(t)$	= $E_0(\boldsymbol{a}_y)$

Left-Hand Circular Polarized (LHCP)

$$\boldsymbol{E}_0 = E_0\left(j\boldsymbol{a}_x - \boldsymbol{a}_y\right) \tag{3}$$

(3) → $$\begin{aligned} \boldsymbol{E}_0(t) &= Re\left[E_0\left(j\boldsymbol{a}_x - \boldsymbol{a}_y\right)e^{j\omega t}\right] \\ &= E_0\left(-\sin\omega t\,\boldsymbol{a}_x - \cos\omega t\,\boldsymbol{a}_y\right) \end{aligned} \tag{4}$$

(4) →

t	=	0	ωt	=	0	$\boldsymbol{E}_0(t)$	= $E_0(-\boldsymbol{a}_y)$
t	=	$\frac{T}{4}$	ωt	=	$\frac{\pi}{2}$	$\boldsymbol{E}_0(t)$	= $E_0(-\boldsymbol{a}_x)$
t	=	$\frac{T}{2}$	ωt	=	π	$\boldsymbol{E}_0(t)$	= $E_0(\boldsymbol{a}_y)$
t	=	$\frac{3T}{4}$	ωt	=	$\frac{3\pi}{2}$	$\boldsymbol{E}_0(t)$	= $E_0(\boldsymbol{a}_x)$
t	=	T	ωt	=	2π	$\boldsymbol{E}_0(t)$	= $E_0(-\boldsymbol{a}_y)$

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Table 4. Derivation of the Far-Field Radiation Integrals and Fields

$$\boldsymbol{E}_A = -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) \tag{1-a}$$

$$\boldsymbol{H}_A = \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A} \tag{1-b}$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{J}$$

Solution to the Inhomogeneous Vector Potential Wave Equation

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_{V'}\boldsymbol{J}(\boldsymbol{r}')\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dv' \tag{1-c}$$

$$\boldsymbol{E}_F = -\frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{F} \tag{2-a}$$

$$\boldsymbol{H}_F = -j\omega\boldsymbol{F} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) \tag{2-b}$$

$$\boldsymbol{\nabla}^2\boldsymbol{F} + k^2\boldsymbol{F} = -\varepsilon\boldsymbol{M}$$

Solution to the Inhomogeneous Vector Potential Wave Equation

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{\varepsilon}{4\pi}\iiint_{V'}\boldsymbol{M}(\boldsymbol{r}')\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dv' \tag{2-c}$$

$$\begin{aligned} R &\equiv |\boldsymbol{r}-\boldsymbol{r}'| \\ &= [(\boldsymbol{r}-\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')]^{\frac{1}{2}} \\ &= [|\boldsymbol{r}|^2 + |\boldsymbol{r}'|^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}']^{\frac{1}{2}} \\ &= \left[r^2 + r'^2 - 2rr'\cos\psi\right]^{\frac{1}{2}} \end{aligned} \tag{3}$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \quad \text{BE}$$

$$\text{(3), BE} \quad \rightarrow \quad R = r\left[1 - \cos\psi\left(\frac{r'}{r}\right) + \frac{1}{2}(\sin\psi)^2\left(\frac{r'}{r}\right)^2 + \frac{1}{2}\cos\psi\,(\sin\psi)^2\left(\frac{r'}{r}\right)^3 + \cdots\right] \tag{4}$$

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Radiation Regions

- Regions are defined by the smallest radius for which the maximum phase error, resulting from an approximation of R, does not exceed $\frac{\pi}{8}$ radians. In other words, if $T(r, r', \psi)$ is the largest discarded term in (4), then $\left(\max_{r',\psi} T\right) k \leq \frac{\pi}{8}$. (5-a)
- D is the largest diameter of the source distribution, i.e., $D \equiv 2r'_{max}$.

Radiating Near-Field (Fresnel) Region

$$R \approx r\left[1 - \cos\psi\left(\frac{r'}{r}\right) + \frac{1}{2}(\sin\psi)^2\left(\frac{r'}{r}\right)^2\right] \tag{5-b}$$

(4), (5-a), (5-b) $\rightarrow$ $$r > 0.62\sqrt{\frac{D^3}{\lambda}}$$

Far-Field (Fraunhofer) Region

$$R \approx r\left[1 - \cos\psi\left(\frac{r'}{r}\right)\right] \tag{5-c}$$

(4), (5-a), (5-c) $\rightarrow$ $$r > \frac{2D^2}{\lambda}$$

Radiation Integrals N and L in the Far-Field Region

In the far-field region, the R in the phase expression of (1-c) and (2-c) is approximated by $R \approx r - r'\cos\psi$ which can be reformulated as $R \approx r - \boldsymbol{a}_r \cdot \boldsymbol{r}'$. Furthermore, the R in their amplitude expression is approximated by $R \approx r$. (6)

(1-c), (6) $\rightarrow$ $$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint_{V'} \boldsymbol{J}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}\,dv' \tag{7}$$

$$\boldsymbol{N}(\boldsymbol{r}) \equiv \iiint_{V'} \boldsymbol{J}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}\,dv' \tag{8}$$

(7), (8) $\rightarrow$ $$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\boldsymbol{N}(\boldsymbol{r}) \tag{9}$$

(2-c), (6) $\rightarrow$ $$\boldsymbol{F}(\boldsymbol{r}) \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\iiint_{V'} \boldsymbol{M}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}\,dv' \tag{10}$$

$$\boldsymbol{L}(\boldsymbol{r}) \equiv \iiint_{V'} \boldsymbol{M}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}\,dv' \tag{11}$$

(10), (11) $\rightarrow$ $$\boldsymbol{F}(\boldsymbol{r}) = \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\boldsymbol{L}(\boldsymbol{r}) \tag{12}$$

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E_A and H_A in the Far-Field Region

$$[\nabla(\nabla\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_r = \frac{\partial^2 A_r}{\partial r^2}+\frac{2}{r}\frac{\partial A_r}{\partial r}-\frac{2}{r^2}A_r + \frac{1}{r}\frac{\partial^2 A_\theta}{\partial r\partial\theta}+\frac{\cos\theta}{r\sin\theta}\frac{\partial A_\theta}{\partial r}-\frac{1}{r^2}\frac{\partial A_\theta}{\partial\theta}-\frac{\cos\theta}{r^2\sin\theta}A_\theta + \frac{1}{r\sin\theta}\frac{\partial^2 A_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi} \quad (13\text{-a})$$

$$[\nabla(\nabla\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_\theta = \frac{1}{r}\frac{\partial^2 A_r}{\partial\theta\partial r}+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\theta}-\frac{1}{r^2(\sin\theta)^2}A_\theta + \frac{1}{r^2\sin\theta}\frac{\partial^2 A_\phi}{\partial\theta\partial\phi}-\frac{\cos\theta}{r^2(\sin\theta)^2}\frac{\partial A_\phi}{\partial\phi} \quad (13\text{-b})$$

$$[\nabla(\nabla\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_\phi = \frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial\phi\partial r}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2 A_\theta}{\partial\phi\partial\theta}+\frac{\cos\theta}{r^2(\sin\theta)^2}\frac{\partial A_\theta}{\partial\phi} + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 A_\phi}{\partial\phi^2} \quad (13\text{-c})$$

$$[\nabla\times\boldsymbol{A}]\cdot\boldsymbol{a}_r = -\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial A_\phi}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}A_\phi \quad (14\text{-a})$$

$$[\nabla\times\boldsymbol{A}]\cdot\boldsymbol{a}_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial A_\phi}{\partial r}-\frac{1}{r}A_\phi \quad (14\text{-b})$$

$$[\nabla\times\boldsymbol{A}]\cdot\boldsymbol{a}_\phi = -\frac{1}{r}\frac{\partial A_r}{\partial\theta}+\frac{\partial A_\theta}{\partial r}+\frac{1}{r}A_\theta \quad (14\text{-c})$$

$$\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right) = -\frac{e^{-jkr}}{r^2}-jk\frac{e^{-jkr}}{r} \quad (15\text{-a})$$

$$\frac{\partial^2}{\partial r^2}\left(\frac{e^{-jkr}}{r}\right) = 2\frac{e^{-jkr}}{r^3}+j2k\frac{e^{-jkr}}{r^2}-k^2\frac{e^{-jkr}}{r} \quad (15\text{-b})$$

In expressions for $\boldsymbol{E}_A$ and $\boldsymbol{H}_A$, neglect terms with $\frac{1}{r^n}$ for $n \geq 2$. (16)

$$\boldsymbol{E}_A\cdot\boldsymbol{a}_r = -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_r - j\frac{1}{\omega\mu\varepsilon}[\nabla(\nabla\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_r \approx -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_r - j\frac{1}{\omega\mu\varepsilon}(-k^2\boldsymbol{A}\cdot\boldsymbol{a}_r) \quad (17\text{-a})$$

(1-a), (9), (13), (15), (16) $\rightarrow$

$$= 0$$

$$\boldsymbol{E}_A\cdot\boldsymbol{a}_\theta = -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_\theta - j\frac{1}{\omega\mu\varepsilon}[\nabla(\nabla\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_\theta \approx -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_\theta \quad (17\text{-b})$$

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$$= -jk\eta\frac{e^{-jkr}}{4\pi r}\boldsymbol{N}\cdot\boldsymbol{a}_\theta$$

$$\boldsymbol{E}_A\cdot\boldsymbol{a}_\phi = -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_\theta - j\frac{1}{\omega\mu\varepsilon}[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})]\cdot\boldsymbol{a}_\theta$$
$$\approx -j\omega\boldsymbol{A}\cdot\boldsymbol{a}_\phi \quad (17\text{-c})$$
$$= -jk\eta\frac{e^{-jkr}}{4\pi r}\boldsymbol{N}\cdot\boldsymbol{a}_\phi$$

$$\boldsymbol{H}_A\cdot\boldsymbol{a}_r = \frac{1}{\mu}[\boldsymbol{\nabla}\times\boldsymbol{A}]\cdot\boldsymbol{a}_r \quad (18\text{-a})$$
$$\approx 0$$

(1-b), (9), (14), (15), (16) →
$$\boldsymbol{H}_A\cdot\boldsymbol{a}_\theta = \frac{1}{\mu}[\boldsymbol{\nabla}\times\boldsymbol{A}]\cdot\boldsymbol{a}_\theta \quad (18\text{-b})$$
$$\approx jk\frac{e^{-jkr}}{4\pi r}\boldsymbol{N}\cdot\boldsymbol{a}_\phi$$

$$\boldsymbol{H}_A\cdot\boldsymbol{a}_\phi = \frac{1}{\mu}[\boldsymbol{\nabla}\times\boldsymbol{A}]\cdot\boldsymbol{a}_\phi \quad (18\text{-c})$$
$$\approx -jk\frac{e^{-jkr}}{4\pi r}\boldsymbol{N}\cdot\boldsymbol{a}_\theta$$

$\boldsymbol{E}_F$ and $\boldsymbol{H}_F$ in the Far-Field Region

Components of $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F})$ have the same expressions as (13) (19)
Components of $\boldsymbol{\nabla}\times\boldsymbol{F}$ have the same expressions as (14) (20)
In expressions for $\boldsymbol{E}_F$ and $\boldsymbol{H}_F$, neglect terms with $\frac{1}{r^n}$ for $n \geq 2$. (21)

$$\boldsymbol{E}_F\cdot\boldsymbol{a}_r = -\frac{1}{\varepsilon}[\boldsymbol{\nabla}\times\boldsymbol{F}]\cdot\boldsymbol{a}_r \quad (22\text{-a})$$
$$\approx 0$$

(2-a), (12), (15), (20), (21) →
$$\boldsymbol{E}_F\cdot\boldsymbol{a}_\theta = -\frac{1}{\varepsilon}[\boldsymbol{\nabla}\times\boldsymbol{F}]\cdot\boldsymbol{a}_\theta \quad (22\text{-b})$$
$$\approx -jk\frac{e^{-jkr}}{4\pi r}\boldsymbol{L}\cdot\boldsymbol{a}_\phi$$

$$\boldsymbol{E}_F\cdot\boldsymbol{a}_\phi = -\frac{1}{\varepsilon}[\boldsymbol{\nabla}\times\boldsymbol{F}]\cdot\boldsymbol{a}_\phi \quad (22\text{-c})$$
$$\approx jk\frac{e^{-jkr}}{4\pi r}\boldsymbol{L}\cdot\boldsymbol{a}_\theta$$

$$\boldsymbol{H}_F\cdot\boldsymbol{a}_r = -j\omega\boldsymbol{F}\cdot\boldsymbol{a}_r - j\frac{1}{\omega\mu\varepsilon}[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F})]\cdot\boldsymbol{a}_r$$

(2-b), (12), (15), (19), (21) →
$$\approx -j\omega\boldsymbol{F}\cdot\boldsymbol{a}_r - j\frac{1}{\omega\mu\varepsilon}(-k^2\boldsymbol{F}\cdot\boldsymbol{a}_r) \quad (23\text{-a})$$
$$= 0$$

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$$\boldsymbol{H}_F \cdot \boldsymbol{a}_\theta = -j\omega \boldsymbol{F} \cdot \boldsymbol{a}_\theta - j\frac{1}{\omega\mu\varepsilon}[\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F})] \cdot \boldsymbol{a}_\theta$$
$$\approx -j\omega \boldsymbol{F} \cdot \boldsymbol{a}_\theta \quad (23\text{-b})$$
$$= -j\frac{k}{\eta}\frac{e^{-jkr}}{4\pi r}\boldsymbol{L} \cdot \boldsymbol{a}_\theta$$

$$\boldsymbol{H}_F \cdot \boldsymbol{a}_\phi = -j\omega \boldsymbol{F} \cdot \boldsymbol{a}_\phi - j\frac{1}{\omega\mu\varepsilon}[\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F})] \cdot \boldsymbol{a}_\phi$$
$$\approx -j\omega \boldsymbol{F} \cdot \boldsymbol{a}_\phi \quad (23\text{-c})$$
$$= -j\frac{k}{\eta}\frac{e^{-jkr}}{4\pi r}\boldsymbol{L} \cdot \boldsymbol{a}_\phi$$

E and H in the Far-Field Region

(17-a), (22-a) →
$$\boldsymbol{E} \cdot \boldsymbol{a}_r = \boldsymbol{E}_A \cdot \boldsymbol{a}_r + \boldsymbol{E}_F \cdot \boldsymbol{a}_r = 0$$

(17-b), (22-b) →
$$\boldsymbol{E} \cdot \boldsymbol{a}_\theta = \boldsymbol{E}_A \cdot \boldsymbol{a}_\theta + \boldsymbol{E}_F \cdot \boldsymbol{a}_\theta = -jk\eta\frac{e^{-jkr}}{4\pi r}\left(\boldsymbol{N} \cdot \boldsymbol{a}_\theta + \frac{1}{\eta}\boldsymbol{L} \cdot \boldsymbol{a}_\phi\right)$$

(17-c), (22-c) →
$$\boldsymbol{E} \cdot \boldsymbol{a}_\phi = \boldsymbol{E}_A \cdot \boldsymbol{a}_\phi + \boldsymbol{E}_F \cdot \boldsymbol{a}_\phi = -jk\eta\frac{e^{-jkr}}{4\pi r}\left(\boldsymbol{N} \cdot \boldsymbol{a}_\phi - \frac{1}{\eta}\boldsymbol{L} \cdot \boldsymbol{a}_\theta\right)$$

(18-a), (23-a)
$$\boldsymbol{H} \cdot \boldsymbol{a}_r = \boldsymbol{H}_A \cdot \boldsymbol{a}_r + \boldsymbol{H}_F \cdot \boldsymbol{a}_r = 0$$

(18-b), (23-b) →
$$\boldsymbol{H} \cdot \boldsymbol{a}_\theta = \boldsymbol{H}_A \cdot \boldsymbol{a}_\theta + \boldsymbol{H}_F \cdot \boldsymbol{a}_\theta = jk\frac{e^{-jkr}}{4\pi r}\left(\boldsymbol{N} \cdot \boldsymbol{a}_\phi - \frac{1}{\eta}\boldsymbol{L} \cdot \boldsymbol{a}_\theta\right)$$

(18-c), (23-c) →
$$\boldsymbol{H} \cdot \boldsymbol{a}_\phi = \boldsymbol{H}_A \cdot \boldsymbol{a}_\phi + \boldsymbol{H}_F \cdot \boldsymbol{a}_\phi = -jk\frac{e^{-jkr}}{4\pi r}\left(\boldsymbol{N} \cdot \boldsymbol{a}_\theta + \frac{1}{\eta}\boldsymbol{L} \cdot \boldsymbol{a}_\phi\right)$$

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Table 5. Derivation of the TEM^z Plane Wave from the TM^z Mode

It is possible to obtain a TEM^z solution from the general TM^z solution by requiring that the z dependance of A_z be of the form e^{-jkz}. With this choice, the E_z component of the TM^z solution will vanish resulting in a TEM^z solution. (1)

Rectangular Coordinates

(1) → $\boldsymbol{A} = A_z \boldsymbol{a}_z$ in which $A_z \equiv A_z(x,y)e^{-jkz}$ (2-a)

$\nabla^2 A_z(x,y) = 0$ (2-b)

Lorentz Gauge

$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -j\omega\mu\varepsilon\phi_e$ (3)

(2-a), (3) → $\phi_e = \dfrac{k}{\omega\mu\varepsilon} A_z$ (4)

(2-b), (4) → $\phi_e \equiv \phi_e(x,y)e^{-jkz}$ (5-a)

$\nabla^2 \phi_e(x,y) = 0$ (5-b)

$\boldsymbol{E}_A, \boldsymbol{H}_A$

$$E_{Ax} = -\frac{k}{\omega\mu\varepsilon}\frac{\partial A_z}{\partial x} = -\frac{\partial \phi_e(x,y)}{\partial x} e^{-jkz}$$

$$E_{Ay} = -\frac{k}{\omega\mu\varepsilon}\frac{\partial A_z}{\partial y} = -\frac{\partial \phi_e(x,y)}{\partial y} e^{-jkz}$$

(4), (5-a) →

$$H_{Ax} = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = \frac{\omega\varepsilon}{k}\frac{\partial \phi_e(x,y)}{\partial y} e^{-jkz}$$

$$H_{Ay} = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} = -\frac{\omega\varepsilon}{k}\frac{\partial \phi_e(x,y)}{\partial x} e^{-jkz}$$

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Cylindrical Coordinates

(1) → $\boldsymbol{A} = A_z \boldsymbol{a}_z$ in which $A_z \equiv A_z(\rho, \phi) e^{-jkz}$ (6-a)

$\nabla^2 A_z(\rho, \phi) = 0$ (6-b)

Lorentz Gauge

$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -j\omega\mu\varepsilon\phi_e$ (7)

(6-a), (7) → $\phi_e = \dfrac{k}{\omega\mu\varepsilon} A_z$ (8)

(6-b), (8) →

$\phi_e \equiv \phi_e(\rho, \phi) e^{-jkz}$ (9-a)

$\nabla^2_{\rho,\phi} \phi_e(\rho, \phi) = 0$ (9-b)

$\boldsymbol{E}_A, \boldsymbol{H}_A$

$$E_{A\rho} = -\frac{k}{\omega\mu\varepsilon}\frac{\partial A_z}{\partial \rho} = -\frac{\partial \phi_e(\rho, \phi)}{\partial \rho} e^{-jkz}$$

$$E_{A\phi} = -\frac{k}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} = -\frac{1}{\rho}\frac{\partial \phi_e(\rho, \phi)}{\partial \phi} e^{-jkz}$$

(8), (9-a) →

$$H_{A\rho} = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} = \frac{\omega\varepsilon}{k}\frac{1}{\rho}\frac{\partial \phi_e(\rho, \phi)}{\partial \phi} e^{-jkz}$$

$$H_{A\phi} = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -\frac{\omega\varepsilon}{k}\frac{\partial \phi_e(\rho, \phi)}{\partial \rho} e^{-jkz}$$

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Table 6. Derivation of the Uniform Plane Wave

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_e \quad (1\text{-a})$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\boldsymbol{M}-j\omega\mu\boldsymbol{H} \quad (1\text{-b})$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\rho_m \quad (1\text{-c})$$
$$\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}+j\omega\varepsilon\boldsymbol{E} \quad (1\text{-d})$$

$$-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{F}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F})=\boldsymbol{\nabla}^2\boldsymbol{F} \quad \text{VI}$$

Conduction (Ohmic) and Dielectric Losses

Conduction losses are taken into account by first splitting the electric current density $\boldsymbol{J}$ into impressed and conduction components, i.e., $\boldsymbol{J}=\boldsymbol{J}_{imp}+\boldsymbol{J}_{cnd}$ and then requiring that $\boldsymbol{J}_{cnd}=\sigma\boldsymbol{E}$ in which σ is the conductivity of the medium. Dielectric losses are taken into account by introducing the complex permittivity, i.e., $\varepsilon=\varepsilon'-j\varepsilon''$ in which $\varepsilon'\equiv\varepsilon_0\varepsilon_r'$ and $\varepsilon''\equiv\varepsilon_0\varepsilon_r''$. (2)
Using the above modifications, equation (1-d) can be rewritten as $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{imp}+j\omega\varepsilon_c\boldsymbol{E}$ in which $\varepsilon_c=\varepsilon'\left(1-j\frac{\sigma_{eff}}{\omega\varepsilon'}\right)$ and $\sigma_{eff}\equiv\sigma+\omega\varepsilon''$.

Source less region of space

$$\rho_e=0,\ \rho_m=0,\ \boldsymbol{M}=\boldsymbol{0},\ \boldsymbol{J}_{imp}=\boldsymbol{0} \quad (3)$$

(1-b), VI-1, (2), (3) $\rightarrow$ $\boldsymbol{\nabla}^2\boldsymbol{E}-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})+k^2\boldsymbol{E}=\boldsymbol{0}$ in which $k^2\equiv\omega^2\mu\varepsilon_c$ (4)

(1-a), (3), (4) $\rightarrow$ $\boldsymbol{\nabla}^2\boldsymbol{E}+k^2\boldsymbol{E}=\boldsymbol{0}$ (5)

$$E_x=E_{0x}e^{-j(k_xx+k_yy+k_zz)} \quad (6\text{-a})$$
(5) $\rightarrow$
$$E_y=E_{0y}e^{-j(k_xx+k_yy+k_zz)} \quad (6\text{-b})$$
$$E_z=E_{0z}e^{-j(k_xx+k_yy+k_zz)} \quad (6\text{-c})$$

$$\boldsymbol{E}=\boldsymbol{E}_0e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

(6) $\rightarrow$
$$\boldsymbol{E}_0\equiv E_{0x}\boldsymbol{a}_x+E_{0y}\boldsymbol{a}_y+E_{0z}\boldsymbol{a}_z \quad (7)$$
$$\boldsymbol{k}\equiv k_x\boldsymbol{a}_x+k_y\boldsymbol{a}_y+k_z\boldsymbol{a}_z \quad \text{in which } k_x^2+k_y^2+k_z^2=k^2$$

(1-a), (3), (7) $\rightarrow$ $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$

(1-b), (3), (7) $\rightarrow$ $\boldsymbol{H}=\frac{1}{\omega\mu}\boldsymbol{k}\times\boldsymbol{E}$

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2- The Transmission Line Model

Table 7. Derivation of the Voltage, Current and Power waves on Transmission Lines

Telegrapher's Equations

Loop KVL

$$-\frac{1}{\Delta z}[v(z+\Delta z,t)-v(z,t)]=L\frac{\partial i(z,t)}{\partial t}+Ri(z,t) \quad (1\text{-a})$$

Node KCL

$$-\frac{1}{\Delta z}[i(z+\Delta z,t)-i(z,t)]=C\frac{\partial v(z+\Delta z,t)}{\partial t}+Gv(z+\Delta z,t) \quad (1\text{-b})$$

$$\Delta z \to 0 \quad (2)$$

(1-a), (2) →
$$-\frac{\partial v(z,t)}{\partial z}=L\frac{\partial i(z,t)}{\partial t}+Ri(z,t) \quad (3\text{-a})$$

(1-b), (2) →
$$-\frac{\partial i(z,t)}{\partial z}=C\frac{\partial v(z,t)}{\partial t}+Gv(z,t) \quad (3\text{-b})$$

Voltage and Current Waves

Time-Harmonic Telegrapher's Equations

(3) →
$$-\frac{\partial V(z)}{\partial z}=(R+j\omega L)I(z) \quad (4\text{-a})$$
$$-\frac{\partial I(z)}{\partial z}=(G+j\omega C)V(z) \quad (4\text{-b})$$

General Solutions

$$V(z)=V^{+}e^{-\gamma z}+V^{-}e^{\gamma z} \quad (5\text{-a})$$
$$I(z)=I^{+}e^{-\gamma z}+I^{-}e^{\gamma z} \quad (5\text{-b})$$

$$I(z)=\frac{V^{+}}{Z_0}e^{-\gamma z}-\frac{V^{-}}{Z_0}e^{\gamma z} \quad (6\text{-a})$$

(4-a), (5-a) →
$$Z_0\equiv\frac{R+j\omega L}{\gamma} \quad (6\text{-b})$$

$$V(z)=I^{+}Z_0e^{-\gamma z}-I^{-}Z_0e^{\gamma z} \quad (7\text{-a})$$

(4-b), (5-b) →
$$Z_0\equiv\frac{\gamma}{G+j\omega C} \quad (7\text{-b})$$

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(6-b), (7-b) → $\gamma^2 = (R + j\omega L)(G + j\omega C)$ (8)

(6-b), (8) → $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 + jX_0$ (9)

The Input Impedance

(5), (6-a) → $Z(z) \equiv \frac{V(z)}{I(z)} = \frac{V^+ e^{-\gamma z} + V^- e^{\gamma z}}{\frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{\gamma z}}$ (10)

Boundary Condition on the Load Side

(10) → $Z(z = 0) = \frac{V^+ + V^-}{\frac{V^+}{Z_0} - \frac{V^-}{Z_0}} = Z_L$ (11)

(11) → $\Gamma_L \equiv \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$ (12)

(10), (12) → $Z(z) = Z_0 \frac{1 + \Gamma_L e^{2\gamma z}}{1 - \Gamma_L e^{2\gamma z}} = Z_0 \frac{Z_L - Z_0 \tanh \gamma z}{Z_0 - Z_L \tanh \gamma z}$ (13)

(13) → $Z_{IN} \equiv Z(z = -l) = Z_0 \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}$ (14)

(12), (14) → $\Gamma_{IN} \equiv \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} = \Gamma_L e^{-2\gamma l}$ (15)

Excited Wave Amplitude

Boundary Condition on the Source Side

$V_S = Z_S I(z = -l) + V(z = -l)$ (16)

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(5), (6-a), (12), (16) →
$$\frac{V^+}{V_S} = \frac{Z_0}{Z_S + Z_0} \frac{1}{1 - \left(\frac{Z_S - Z_0}{Z_S + Z_0}\right)(\Gamma_L e^{-2\gamma l})} e^{-\gamma l} \quad (17)$$

$$\Gamma_S \equiv \frac{Z_S - Z_0}{Z_S + Z_0} \quad (18)$$

(15), (17), (18) →
$$\frac{V^+}{V_S} = \frac{1}{2}\left(\frac{1 - \Gamma_S}{1 - \Gamma_S \Gamma_{IN}}\right) e^{-\gamma l} \quad (19)$$

Power Waves

(5), (6-a), (12) →
$$V(z) = V^+(1 + \Gamma_L e^{2\gamma z})e^{-\gamma z} \quad (20\text{-a})$$
$$I(z) = \frac{V^+}{Z_0}(1 - \Gamma_L e^{2\gamma z})e^{-\gamma z} \quad (20\text{-b})$$

Incident Power Wave

(20) →
$$\begin{aligned} a(z) &\equiv \frac{1}{2\sqrt{R_0}}\big(V(z) + Z_0 I(z)\big) \\ &= \frac{V^+}{\sqrt{R_0}} e^{-\gamma z} \end{aligned} \quad (21\text{-a})$$

Reflected Power Wave

(20) →
$$\begin{aligned} b(z) &\equiv \frac{1}{2\sqrt{R_0}}\big(V(z) - Z_0^* I(z)\big) \\ &= \frac{V^+}{\sqrt{R_0}} e^{-\gamma z}\left[\frac{R_0}{Z_0}\left(j\frac{X_0}{R_0} + \Gamma_L e^{2\gamma z}\right)\right] \end{aligned} \quad (21\text{-b})$$

(21) →
$$V(z) = \frac{1}{\sqrt{R_0}}\big(Z_0^* a(z) + Z_0 b(z)\big) \quad (22\text{-a})$$
$$I(z) = \frac{1}{\sqrt{R_0}}\big(a(z) - b(z)\big) \quad (22\text{-a})$$

(21), (22) →
$$\begin{aligned} P_{av}(z) &\equiv Re\left(\frac{1}{2}V(z)I^*(z)\right) \\ &= \frac{1}{2}(|a(z)|^2 - |b(z)|^2) \\ &= \frac{1}{2}\left|\frac{V^+}{\sqrt{R_0}}\right|^2 \left(1 - \left(\frac{R_0}{|Z_0|}\right)^2 \left|j\frac{X_0}{R_0} + \Gamma_L e^{2\gamma z}\right|^2\right) \end{aligned}$$

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Table 8. Derivation of the Impedance and Admittance Smith Charts

Impedance Smith Chart

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{Z_L^n - 1}{Z_L^n + 1} \qquad Z_L^n \equiv \frac{Z_L}{Z_0}\end{aligned} \tag{1}$$

(1) → $$\begin{aligned}Z_L^n &= \frac{1 + \Gamma_L}{1 - \Gamma_L} \\ &= r + jx\end{aligned} \tag{2}$$

(2) → $$r = \frac{1 - \Gamma_{Lr}^2 - \Gamma_{Li}^2}{(1 - \Gamma_{Lr})^2 + \Gamma_{Li}^2} \tag{3-a}$$

$$x = \frac{2\Gamma_{Li}}{(1 - \Gamma_{Lr})^2 + \Gamma_{Li}^2} \tag{3-b}$$

(3-a) → $$\left(\Gamma_{Lr} - \frac{r}{r+1}\right)^2 + \Gamma_{Li}^2 = \left(\frac{1}{r+1}\right)^2$$

(3-b) → $$(\Gamma_{Lr} - 1)^2 + \left(\Gamma_{Li} - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

Admittance Smith Chart

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= -\frac{Y_L - Y_0}{Y_L + Y_0} \\ &= \frac{Y_L^n - 1}{Y_L^n + 1} \qquad Y_L^n \equiv \frac{Y_L}{Y_0}\end{aligned} \tag{4}$$

(4) → $$\begin{aligned}Y_L^n &= \frac{1 - \Gamma_L}{1 + \Gamma_L} \\ &= g + jb\end{aligned} \tag{5}$$

(5) → $$g = \frac{1 - \Gamma_{Lr}^2 - \Gamma_{Li}^2}{(1 + \Gamma_{Lr})^2 + \Gamma_{Li}^2} \tag{6-a}$$

$$b = -\frac{2\Gamma_{Li}}{(1 + \Gamma_{Lr})^2 + \Gamma_{Li}^2} \tag{6-b}$$

(6-a) → $$\left(\Gamma_{Lr} + \frac{g}{g+1}\right)^2 + \Gamma_{Li}^2 = \left(\frac{1}{g+1}\right)^2$$

(6-b) → $$(\Gamma_{Lr} + 1)^2 + \left(\Gamma_{Li} + \frac{1}{b}\right)^2 = \left(\frac{1}{b}\right)^2$$

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3- Network Analysis

Table 9. Derivation of the Scattering Matrix of a Segment of Transmission Line

Direction Convention

At each port:

- $z = 0$ at the terminals.
- $z > 0$ looking towards the network.
- $z < 0$ looking away from the network.

(1)

Therefore, the port's excitation can be expressed as $a(z) = V_n e^{-j\gamma z}$.

$$S_{11} \equiv \frac{V_{n,1}^-}{V_{n,1}^+}\left(V_{n,2}^+ = 0\right) \quad (2)$$

(1) → $V_{n,1}^+ = V_n$ (3-a)

(1), (2) → $V_{n,1}^- = 0$ (3-b)

(2), (3) → $S_{11} = 0$

$$S_{12} \equiv \frac{V_{n,1}^-}{V_{n,2}^+}\left(V_{n,1}^+ = 0\right) \quad (4)$$

(1) → $V_{n,2}^+ = V_n$ (5-a)

(1), (4) → $V_{n,1}^- = V_n e^{-\gamma l}$ (5-b)

(4), (5) → $S_{12} = e^{-\gamma l}$

$$S_{21} \equiv \frac{V_{n,2}^-}{V_{n,1}^+}\left(V_{n,2}^+ = 0\right) \quad (6)$$

(1) → $V_{n,1}^+ = V_n$ (7-a)

(1), (6) → $V_{n,2}^- = V_n e^{-\gamma l}$ (7-b)

(6), (7) → $S_{21} = e^{-\gamma l}$

$$S_{22} \equiv \frac{V_{n,2}^-}{V_{n,2}^+}\left(V_{n,1}^+ = 0\right) \quad (8)$$

(1) → $V_{n,2}^+ = V_n$ (9-a)

(1), (8) → $V_{n,2}^- = 0$ (9-b)

(8), (9) → $S_{22} = 0$

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Table 10. Derivation of the Two-Port Power Relations

$$V_{n,1}^{-} = S_{11}V_{n,1}^{+} + S_{12}V_{n,2}^{+} \quad (1\text{-a})$$

$$V_{n,2}^{-} = S_{21}V_{n,1}^{+} + S_{22}V_{n,2}^{+} \quad (1\text{-b})$$

$$\Gamma_L \equiv \frac{V_{n,2}^{+}}{V_{n,2}^{-}} \quad (2\text{-a})$$

$$\Gamma_S \equiv \frac{V_{n,1}^{+}}{V_{n,1}^{-}} \quad (2\text{-b})$$

$$\frac{|V_{n,1}^{+}|}{|V_{Sn}|} = \frac{\sqrt{Z_S}}{\sqrt{Z_0^{Port}}}\left(\frac{1}{2}\frac{|1-\Gamma_S|}{|1-\Gamma_S\Gamma_{IN}|}\right) \quad (3\text{-a})$$

$$\frac{|V_{n,2}^{-}|}{|V_{Sn}^{equiv}|} = \frac{\sqrt{Z_0^{Port}}}{\sqrt{Z_0^{Load}}}\left(\frac{1}{2}\frac{|1-\Gamma_{OUT}|}{|1-\Gamma_{OUT}\Gamma_L|}\right) \quad (3\text{-b})$$

Input Reflection Coefficient

$$\Gamma_{IN} \equiv \frac{V_{n,1}^{-}}{V_{n,1}^{+}} \quad (4)$$

(1-b), (2-a) $\rightarrow$ $$V_{n,2}^{+} = \frac{S_{21}\Gamma_L}{1-S_{22}\Gamma_L}V_{n,1}^{+} \quad (5)$$

(1-a), (4), (5) $\rightarrow$ $$\Gamma_{IN} = \frac{S_{11}-\Delta\Gamma_L}{1-S_{22}\Gamma_L} \qquad \Delta \equiv S_{11}S_{22} - S_{12}S_{21} \quad (6)$$

Output Reflection Coefficient

$$\Gamma_{OUT} \equiv \frac{V_{n,2}^{-}}{V_{n,2}^{+}} \quad (7)$$

(1-a), (2-b) $\rightarrow$ $$V_{n,1}^{+} = \frac{S_{12}\Gamma_S}{1-S_{11}\Gamma_S}V_{n,2}^{+} \quad (8)$$

(1-b), (7), (8) $\rightarrow$ $$\Gamma_{OUT} = \frac{S_{22}-\Delta\Gamma_S}{1-S_{11}\Gamma_S} \qquad \Delta \equiv S_{11}S_{22} - S_{12}S_{21} \quad (9)$$

Source Mismatch Factor

In the following derivations P_{IN} is used to denote the power entering port one, i.e., $P_1^{entering}$ and P_{AVS} is used to denote the power that is available from the source, i.e., $P_1^{available}$.

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$$M_S \equiv \frac{P_{IN}}{P_{AVS}} \tag{10}$$

$$P_{IN} = \frac{1}{2}\left|V_{n,1}^{+}\right|^2(1-|\Gamma_{IN}|^2) \tag{11-a}$$

(3-a), (11-a) →
$$P_{IN} = \frac{1}{8}|V_{Sn}|^2\frac{Z_S}{Z_0^{Port}}\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_{IN}|^2}(1-|\Gamma_{IN}|^2) \tag{11-b}$$

(11-b) →
$$\begin{aligned} P_{AVS} &\equiv P_{IN}(\Gamma_{IN}=\Gamma_S^*) \\ &= \frac{1}{8}|V_{Sn}|^2\frac{Z_S}{Z_0^{Port}}\frac{|1-\Gamma_S|^2}{1-|\Gamma_S|^2} \end{aligned} \tag{12}$$

(10), (11-b), (12) →
$$M_S = \frac{(1-|\Gamma_S|^2)(1-|\Gamma_{IN}|^2)}{|1-\Gamma_S\Gamma_{IN}|^2} \tag{13}$$

Load Mismatch Factor

In the following derivations P_L is used to denote the power exiting port two, i.e., $P_2^{exiting}$ and P_{AVN} is used to denote the power that is available from the network, i.e., $P_2^{available}$.

$$M_L \equiv \frac{P_L}{P_{AVN}} \tag{14}$$

$$P_L = \frac{1}{2}\left|V_{n,2}^{-}\right|^2(1-|\Gamma_L|^2) \tag{15-a}$$

(3-b), (15-a) →
$$P_L = \frac{1}{8}\left|V_{Sn}^{equiv}\right|^2\frac{Z_0^{Port}}{Z_0^{Load}}\frac{|1-\Gamma_{OUT}|^2}{|1-\Gamma_{OUT}\Gamma_L|^2}(1-|\Gamma_L|^2) \tag{15-b}$$

(15-b) →
$$\begin{aligned} P_{AVN} &\equiv P_L(\Gamma_L=\Gamma_{OUT}^*) \\ &= \frac{1}{8}\left|V_{Sn}^{equiv}\right|^2\frac{Z_0^{Port}}{Z_0^{Load}}\frac{|1-\Gamma_{OUT}|^2}{1-|\Gamma_{OUT}|^2} \end{aligned} \tag{16}$$

(14), (15-b), (16) →
$$M_L = \frac{(1-|\Gamma_L|^2)(1-|\Gamma_{OUT}|^2)}{|1-\Gamma_{OUT}\Gamma_L|^2} \tag{17}$$

Operating Power Gain

(1-b), (5), (11-a), (15-a) →
$$\begin{aligned} G_P &\equiv \frac{P_L}{P_{IN}} \\ &= \frac{1}{1-|\Gamma_{IN}|^2}|S_{21}|^2\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2} \end{aligned} \tag{18}$$

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An Important Identity

(6) → $\Delta\Gamma_L = S_{11} - \Gamma_{IN}(1 - S_{22}\Gamma_L)$ (19-a)

(9) → $\Delta\Gamma_S = S_{22} - \Gamma_{OUT}(1 - S_{11}\Gamma_S)$ (19-b)

(19) → $$\frac{\Gamma_L}{\Gamma_S} = \frac{S_{11} - \Gamma_{IN}(1 - S_{22}\Gamma_L)}{S_{22} - \Gamma_{OUT}(1 - S_{11}\Gamma_S)} \quad (20)$$

(20) → $$(1 - S_{11}\Gamma_S)(1 - \Gamma_L\Gamma_{OUT}) = (1 - S_{22}\Gamma_L)(1 - \Gamma_S\Gamma_{IN}) \quad (21)$$

Available Power Gain

$$G_A \equiv \frac{P_{AVN}}{P_{AVS}} \quad (22)$$

(10), (14), (22) → $$G_A = \frac{M_S}{M_L}\frac{P_L}{P_{IN}} \quad (23)$$

(13), (17), (18), (21), (23) → $$G_A = \frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2}|S_{21}|^2\frac{1}{1 - |\Gamma_{OUT}|^2}$$

Transducer Power Gain

$$G_T \equiv \frac{P_L}{P_{AVS}} \quad (24)$$

(10), (24) → $$G_T = M_S\frac{P_L}{P_{IN}} \quad (25)$$

(13), (18), (21), (25) → $$G_T = \frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2}|S_{21}|^2\frac{1 - |\Gamma_L|^2}{|1 - \Gamma_L\Gamma_{OUT}|^2}$$

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Table 11. Derivation of the Scattering Matrix of the Circulator

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \quad (1)$$

Circulator is a Non-Reciprocal, Matched and Lossless three-port network. (2)

Reciprocal Network
$$\boldsymbol{S}^T = \boldsymbol{S} \quad (3\text{-a})$$
Matched Network
$$S_{ii} = 0 \quad (3\text{-b})$$
Lossless Network
$$\boldsymbol{S}^T \boldsymbol{S}^* = \boldsymbol{I} \quad (3\text{-c})$$

(1), (2), (3-a), (3-b) $\rightarrow$
$$\boldsymbol{S} = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix} \quad (4)$$

(3-c), (4) $\rightarrow$
$$|S_{21}|^2 + |S_{31}|^2 = 1$$
$$|S_{12}|^2 + |S_{32}|^2 = 1$$
$$|S_{13}|^2 + |S_{23}|^2 = 1$$
$$S_{31}S_{32}^* = 0$$
$$S_{21}S_{23}^* = 0$$
$$S_{12}S_{13}^* = 0$$

Table 12. Derivation of the Scattering Matrix of the Directional Coupler

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \quad (1)$$

Directional Coupler is Reciprocal, Matched and Lossless four-port network in which $|S_{32}| = |S_{41}| = 0$. (2)

Reciprocal Network
$$\boldsymbol{S}^T = \boldsymbol{S} \quad (3\text{-a})$$
Matched Network
$$S_{ii} = 0 \quad (3\text{-b})$$
Lossless Network
$$\boldsymbol{S}^T \boldsymbol{S}^* = \boldsymbol{I} \quad (3\text{-c})$$

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(1), (2), (3-a), (3-b) $\rightarrow$ $$\boldsymbol{S} = \begin{bmatrix} 0 & S_{21} & S_{31} & 0 \\ S_{21} & 0 & 0 & S_{42} \\ S_{31} & 0 & 0 & S_{43} \\ 0 & S_{42} & S_{43} & 0 \end{bmatrix} \quad (4)$$

(3-c), (4) $\rightarrow$

$$|S_{21}|^2 + |S_{31}|^2 = 1 \quad (5\text{-a})$$
$$|S_{21}|^2 + |S_{42}|^2 = 1 \quad (5\text{-b})$$
$$|S_{31}|^2 + |S_{43}|^2 = 1 \quad (5\text{-c})$$
$$|S_{42}|^2 + |S_{43}|^2 = 1 \quad (5\text{-d})$$
$$S_{21}S_{42}^* + S_{31}S_{43}^* = 0 \quad (5\text{-e})$$
$$S_{21}S_{31}^* + S_{42}S_{43}^* = 0 \quad (5\text{-f})$$

(5-a), (5-b) $\rightarrow$ $|S_{31}| = |S_{42}|$ (6-a)

(5-a), (5-c) $\rightarrow$ $|S_{21}| = |S_{43}|$ (6-b)

$$S_{21} \equiv \alpha \quad (7\text{-a})$$
$$S_{31} \equiv \beta e^{j\theta} \quad (7\text{-b})$$
$$S_{42} \equiv \beta e^{j\phi} \quad (7\text{-c})$$

(6-b), (7-a) $\rightarrow$ $S_{43} \equiv \alpha$ (7-d)

(5-e), (7) $\rightarrow$ $\theta + \phi = \pm\pi$

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4- Nonlinearity, Noise and Dynamic Range

Table 13. Derivation of the Nonlinearity measures P1dB and IP3

$$v_i = V_0 \cos \omega_1 t + V_0 \cos \omega_2 t \quad (1)$$

Input Power of the Input Frequency ω_1

$$(1) \rightarrow P_{IN} = \frac{1}{2} V_0^2 \quad (2)$$

Output Power of the Input Frequency ω_1

Nonlinear Output Power $P_{\omega_1}^{nonlinear}$

$$P_{\omega_1}^{nonlinear} \approx \frac{1}{2}\left(a_1 V_0 + \frac{9}{4} a_3 V_0^3\right)^2 \quad (3)$$

$$(2), (3) \rightarrow \begin{aligned} P_{\omega_1}^{nonlinear} &\approx \frac{1}{2}\left(\sqrt{2} a_1 \sqrt{P_{IN}} + \frac{9}{2}\sqrt{2} a_3 \sqrt{P_{IN}} P_{IN}\right)^2 \\ &= a_1^2 P_{IN} + 9 a_1 a_3 P_{IN}^2 + \frac{81}{4} a_3^2 P_{IN}^3 \end{aligned} \quad (4)$$

Linearized Output Power P_{ω_1}

$$P_{\omega_1} \equiv a_1^2 P_{IN} \quad (5)$$

The Limit of Linearized Output Power of ω_1: P1dB

When P_{ω_1} has diverged 1^{dB} from $P_{\omega_1}^{nonlinear}$ it is no longer a good approximation of the latter. The P_{IN} for which $\frac{P_{\omega_1}^{nonlinear}}{P_{\omega_1}} = \alpha$ (where $10 \log_{10} \alpha = -1$), represents the maximum input power for which the linearized output power P_{ω_1} is a useful approximation. (6)

$$(4), (5), (6) \rightarrow \frac{a_1^2 P_{IN} + 9 a_1 a_3 P_{IN}^2 + \frac{81}{4} a_3^2 P_{IN}^3}{a_1^2 P_{IN}} = \alpha \quad (7)$$

$$(7) \rightarrow P_{IN}^2 - \frac{4}{9}\frac{|a_1|}{|a_3|} P_{IN} + \frac{4}{81}\left(\frac{a_1}{a_3}\right)^2 (1 - \alpha) = 0 \quad \text{and } a_3 < 0 \quad (8)$$

$$(8) \rightarrow \begin{aligned} P_{IN}^{P1dB} &= \frac{2}{9}\frac{|a_1|}{|a_3|}\left(1 - \sqrt{\alpha}\right) \\ &\equiv \text{IP1dB} \end{aligned} \quad (9)$$

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Output Power of the Third Order Intermodulation Frequency $2\omega_1 - \omega_2$

$$P_{2\omega_1-\omega_2} \approx \frac{1}{2}\left(\frac{3}{4}a_3 V_0^3\right)^2 \tag{10}$$

(2), (10) →
$$\begin{aligned} P_{2\omega_1-\omega_2} &\approx \frac{1}{2}\left(\frac{3\sqrt{2}}{2}a_3\sqrt{P_{IN}}P_{IN}\right)^2 \\ &= \frac{9}{4}a_3^2 P_{IN}^3 \end{aligned} \tag{11}$$

Third Order Intercept Point: IP3

Input IP3 is the input power for which $P_{2\omega_1-\omega_2} = P_{\omega_1}$. (12)

(5), (11), (12) →
$$\begin{aligned} P_{IN}^{IP3} &= \frac{2}{3}\frac{|a_1|}{|a_3|} \\ &\equiv \text{IIP3} \end{aligned} \tag{13-a}$$

(5), (12), (13-a) →
$$\begin{aligned} P_{\omega_1}^{IP3} &= \frac{2}{3}\frac{|a_1|^3}{|a_3|} \\ &= P_{2\omega_1-\omega_2}^{IP3} \\ &\equiv \text{OIP3} \end{aligned} \tag{13-b}$$

IP3 as Link between P_{ω_1} and $P_{2\omega_1-\omega_2}$

(5), (11), (13-b) →
$$\begin{aligned} P_{2\omega_1-\omega_2} &= \left(\frac{3}{2}\frac{|a_3|}{|a_1|^3}\right)^2 P_{\omega_1}^3 \\ &= \frac{1}{OIP3^2}P_{\omega_1}^3 \end{aligned} \tag{14}$$

IP3 and P1dB

(9), (13-a) → $P_{IN}^{P1dB} = \frac{1-\sqrt{\alpha}}{3}P_{IN}^{IP3}$

IP3 of Cascaded Systems

Coherent Intermodulation Products

(14) → At the output of stage one, the third-order intermodulation $2\omega_1 - \omega_2$ voltage resulting from its input ω_1 is given by (prime denotes attributes of the first stage): (15-a)

$$V'_{n,2\omega_1-\omega_2} = \frac{\sqrt{2P'^{\,3}_{\omega_1}}}{OIP3'}$$

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(14) → At the output of stage two, the third-order intermodulation $2\omega_1 - \omega_2$ voltage resulting from its input ω_1 is given by (double-prime denotes attributes of the second stage): (15-b)

$$V''_{n,2\omega_1-\omega_2} = \frac{\sqrt{2P''^{\,3}_{\omega_1}}}{OIP3''}$$

(15-a), (15-b) → At the output of stage two, the total third-order intermodulation $2\omega_1 - \omega_2$ voltage is given by: (15-c)

$$V_{n,2\omega_1-\omega_2} = \sqrt{G''}\frac{\sqrt{2P'^{\,3}_{\omega_1}}}{OIP3'} + \frac{\sqrt{2P''^{\,3}_{\omega_1}}}{OIP3''}$$

(15-c) → $$V_{n,2\omega_1-\omega_2} = \left[\frac{1}{G''OIP3'} + \frac{1}{OIP3''}\right]\sqrt{2P''^{\,3}_{\omega_1}} \quad (16)$$

(16) → $$P_{2\omega_1-\omega_2} = \left[\frac{1}{G''OIP3'} + \frac{1}{OIP3''}\right]^2 P''^{\,3}_{\omega_1} \quad (17)$$

(14), (17) → $$OIP3 = \left[\frac{1}{G''OIP3'} + \frac{1}{OIP3''}\right]^{-1}$$

Non-Coherent Intermodulation Products

(14) → At the output of stage one, the third-order intermodulation $2\omega_1 - \omega_2$ power resulting from its input ω_1 is given by (prime denotes attributes of the first stage): (18-a)

$$P'_{2\omega_1-\omega_2} = \frac{1}{OIP3'^2}P'^{\,3}_{\omega_1}$$

(14) → At the output of stage two, the third-order intermodulation $2\omega_1 - \omega_2$ power resulting from its input ω_1 is given by (double-prime denotes attributes of the second stage): (18-b)

$$P''_{2\omega_1-\omega_2} = \frac{1}{OIP3''^2}P''^{\,3}_{\omega_1}$$

(18-a), (18-b) → At the output of stage two, the total third-order intermodulation $2\omega_1 - \omega_2$ power is given by: (18-c)

$$P_{2\omega_1-\omega_2} = G''\frac{1}{OIP3'^2}P'^{\,3}_{\omega_1} + \frac{1}{OIP3''^2}P''^{\,3}_{\omega_1}$$

(18-c) → $$P_{2\omega_1-\omega_2} = \left[\frac{1}{G''^2OIP3'^2} + \frac{1}{OIP3''^2}\right]P''^{\,3}_{\omega_1} \quad (19)$$

(14), (19) → $$OIP3 = \left[\frac{1}{G''^2OIP3'^2} + \frac{1}{OIP3''^2}\right]^{-\frac{1}{2}}$$

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Table 14. Derivation of the Noise Figure and Equivalent Temperature of Two-Ports

Noise Figure of a Two-Port Network

$$F \equiv \frac{S_{IN}/N_0}{S_{OUT}/N_{OUT}} = \frac{S_{IN}}{S_{OUT}}\frac{N_{OUT}}{N_0} \quad (1)$$

$$\begin{aligned} N_{OUT} &= G_A N_0 + G_A N_{network} \\ &= G_A k T_0 B + G_A k T_e B \\ &= G_A k (T_0 + T_e) B \end{aligned} \quad (2)$$

(1), (2) $\rightarrow$ $F = 1 + \frac{T_e}{T_0}$ (3)

Noise Figure of a Cascade of Two-Ports

$$N_{OUT,1} = G_{A,1} N_0 + G_{A,1} N_1 \quad (4\text{-a})$$
$$N_{OUT,2} = G_{A,2} N_{OUT,1} + G_{A,2} N_2 \quad (4\text{-b})$$
$$N_{OUT,3} = G_{A,3} N_{OUT,2} + G_{A,3} N_3 \quad (4\text{-c})$$

$$\begin{aligned} N_{OUT,3} &= G_{A,1}G_{A,2}G_{A,3}N_0 + G_{A,1}G_{A,2}G_{A,3}N_1 + G_{A,2}G_{A,3}N_2 + G_{A,3}N_3 \\ &= G_{A,1}G_{A,2}G_{A,3}kT_0B + G_{A,1}G_{A,2}G_{A,3}k\left(T_{e,1} + \frac{1}{G_{A,1}}T_{e,2} + \frac{1}{G_{A,1}G_{A,2}}T_{e,3}\right)B \\ &= G_{A,1}G_{A,2}G_{A,3}k(T_0 + T_e)B \end{aligned}$$

(4) $\rightarrow$ (5)

(1), (5) $\rightarrow$ $F = F_1 + \frac{F_2 - 1}{G_{A,1}} + \frac{F_3 - 1}{G_{A,1}G_{A,2}}$

Equivalent Temperature of a Low-loss TL

Equivalent Temperature of a Passive Two-Port Network

The available noise power at the input is at temperature $T^{(^\circ K)}$. The network is in thermal equilibrium with its environment and, therefore, is at temperature $T^{(^\circ K)}$. As a result, the available noise power at the output is also at temperature $T^{(^\circ K)}$. (7)

Since the available noise power at the output is given by $N_{OUT} = G_A N_{IN} + G_A N_{network}$, from (7) it follows that the contribution of the network to the available noise power at the output is given by $N_{network} = k\left(\left(\frac{1-G_A}{G_A}\right)T\right)B$. (8)

Therefore, the network's equivalent temperature is given by $T_e \equiv \left(\frac{1-G_A}{G_A}\right)T$.

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Available Gain of a Segment of Low-loss TL

$$\boldsymbol{S} = \begin{bmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{bmatrix} \tag{9}$$

$$L \equiv e^{2\alpha l} \tag{10}$$

(9), (10) → $$\boldsymbol{S} = \begin{bmatrix} 0 & \frac{1}{\sqrt{L}} e^{-j\beta l} \\ \frac{1}{\sqrt{L}} e^{-j\beta l} & 0 \end{bmatrix} \tag{11}$$

$$\Gamma_{OUT} = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S} \quad \Delta \equiv S_{11}S_{22} - S_{12}S_{21} \tag{12}$$

(11), (12) → $$\Gamma_{OUT} = \frac{1}{L} e^{-j2\beta l}\Gamma_S \tag{13}$$

$$G_A = \frac{1 - |\Gamma_S|^2}{|1 - S_{11}\Gamma_S|^2} |S_{21}|^2 \frac{1}{1 - |\Gamma_{OUT}|^2} \tag{14}$$

(11), (13), (14) → $$G_A = \frac{L(1 - |\Gamma_S|^2)}{L^2 - |\Gamma_S|^2} \tag{15}$$

The Equivalent Temperature

(3), (8), (15) → $$T_e = \frac{(L-1)\left(1 + \frac{1}{L}|\Gamma_S|^2\right)}{1 - |\Gamma_S|^2} T$$

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Part II

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5- Radiating Elements and Arrays

Table 15. Far-field Radiation of Some Hypothetical Source Distributions

Far-Field Radiation Integrals

$$\boldsymbol{N}(\boldsymbol{r}) = \iiint_{V'} \boldsymbol{J}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}dv' \tag{1-a}$$

$$\boldsymbol{L}(\boldsymbol{r}) = \iiint_{V'} \boldsymbol{M}(\boldsymbol{r}')e^{jk\boldsymbol{a}_r\cdot\boldsymbol{r}'}dv' \tag{1-b}$$

Useful Relations

Fourier Transform of the Rectangular (Pi) Function

$$\int_{-\infty}^{+\infty} \Pi(t)e^{-j2\pi ft}dt = sinc(\pi f) \tag{UR-I}$$

$$\int e^{\alpha x}\sin(\beta x+\gamma)\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha\sin(\beta x+\gamma) - \beta\cos(\beta x+\gamma)] \tag{UR-II}$$

$$\int_0^{2\pi} \cos n\phi' \sin\Delta\phi\, e^{jx\cos\Delta\phi}d\phi' = 2\pi n(j)^{n+1}\sin n\phi\frac{J_n(x)}{x} \tag{UR-III}$$

$$\int_0^{2\pi} \cos n\phi' \cos\Delta\phi\, e^{jx\cos\Delta\phi}d\phi' = -2\pi(j)^{n+1}\cos n\phi\, J_n'(x) \tag{UR-IV}$$

$$\int_0^{2\pi} e^{jn\phi}e^{jx\cos\phi}d\phi = 2\pi(j)^n J_n(x) \tag{UR-V}$$

$$\int xJ_0(x)dx = xJ_1(x) \tag{UR-VI}$$

Impulse of Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\delta(z)\boldsymbol{a}_z \tag{2-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{2-b}$$

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$$
\begin{aligned}
\boldsymbol{N}(\boldsymbol{r}) &= \iiint_{V'} [I_0\delta(x')\delta(y')\delta(z')\boldsymbol{a}_z] e^{jk\boldsymbol{a}_r\cdot[z'\boldsymbol{a}_z]} dv' \\
\text{(1-a), (2)} \rightarrow \quad &= I_0 \left(\iiint_{V'} \delta(x')\delta(y')\delta(z') e^{jkz'\cos\theta} dx'dy'dz' \right) \boldsymbol{a}_z \\
&= I_0 \left(\left[\int \delta(x')dx' \right] \left[\int \delta(y')dy' \right] \left[\int \delta(z') e^{jkz'\cos\theta} dz' \right] \right) \boldsymbol{a}_z \\
&= I_0 \boldsymbol{a}_z
\end{aligned}
$$

Linear Constant Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z \tag{3-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{3-b}$$

$$
\begin{aligned}
\boldsymbol{N}(\boldsymbol{r}) &= \iiint_{V'} \left[I_0\delta(x')\delta(y')\Pi\left(\frac{z'}{l}\right)\boldsymbol{a}_z \right] e^{jk\boldsymbol{a}_r\cdot[z'\boldsymbol{a}_z]} dv' \\
&= I_0 \left(\iiint_{V'} \delta(x')\delta(y')\Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dx'dy'dz' \right) \boldsymbol{a}_z \\
\text{UR-I, (1-a), (3)} \rightarrow \quad &= I_0 \left(\left[\int \delta(x')dx' \right] \left[\int \delta(y')dy' \right] \left[\int \Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dz' \right] \right) \boldsymbol{a}_z \\
&= I_0 \left(\int \Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dz' \right) \boldsymbol{a}_z \\
&= I_0 l \left(\int \Pi(u) e^{-j2\pi\left(-\frac{kl}{2\pi}\cos\theta\right)u} du \right) \boldsymbol{a}_z \\
&= I_0 l \left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} \right] \boldsymbol{a}_z
\end{aligned}
$$

Linear Sinusoidal Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\sin\left(k\left(\frac{l}{2} - |z|\right)\right)\boldsymbol{a}_z \quad \text{and } |z| \le \frac{l}{2} \tag{4-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{4-b}$$

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$$\boldsymbol{N}(\boldsymbol{r}) = \iiint_{V'} \left[I_0 \delta(x')\delta(y') \sin\left(k\left(\frac{l}{2} - |z'|\right)\right) \boldsymbol{a}_z \right] e^{jk\boldsymbol{a}_r \cdot [z'\boldsymbol{a}_z]} dv'$$

$$= I_0 \left(\iiint_{V'} \delta(x')\delta(y') \sin\left(k\left(\frac{l}{2} - |z'|\right)\right) e^{jkz' \cos\theta} dx'dy'dz' \right) \boldsymbol{a}_z$$

(1-a), (4) →

$$= I_0 \left(\left[\int \delta(x')dx' \right] \left[\int \delta(y')dy' \right] \left[\int_{-\frac{l}{2}}^{+\frac{l}{2}} \sin\left(k\left(\frac{l}{2} - |z'|\right)\right) e^{jkz' \cos\theta} dz' \right] \right) \boldsymbol{a}_z$$

$$= I_0 \left(\int_{-\frac{l}{2}}^{+\frac{l}{2}} \sin\left(k\left(\frac{l}{2} - |z'|\right)\right) e^{jkz' \cos\theta} dz' \right) \boldsymbol{a}_z \quad (5)$$

Simplification of Integrals

$$\int \sin\left(k\left(\frac{l}{2} + z'\right)\right) e^{jkz' \cos\theta} dz'$$

$$= -\frac{e^{jkz' \cos\theta}}{k(\sin\theta)^2} \left[\cos\left(k\left(\frac{l}{2} + z'\right)\right) - j\cos\theta \sin\left(k\left(\frac{l}{2} + z'\right)\right)\right] \quad (6\text{-a})$$

UR-II, (5) →

$$\int \sin\left(k\left(\frac{l}{2} - z'\right)\right) e^{jkz' \cos\theta} dz'$$

$$= \frac{e^{jkz' \cos\theta}}{k(\sin\theta)^2} \left[\cos\left(k\left(\frac{l}{2} - z'\right)\right) + j\cos\theta \sin\left(k\left(\frac{l}{2} - z'\right)\right)\right] \quad (6\text{-b})$$

(5), (6) →

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}} \sin\left(k\left(\frac{l}{2} - |z'|\right)\right) e^{jkz' \cos\theta} dz' \quad (7)$$

$$= \frac{2}{k(\sin\theta)^2} \left(\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right)$$

(5), (7) →

$$\boldsymbol{N}(\boldsymbol{r}) = \frac{2}{k} I_0 \left(\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{(\sin\theta)^2} \right) \boldsymbol{a}_z$$

Circular (Loop) Constant Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0 \delta(\rho - \rho_0)\delta(z)\boldsymbol{a}_\phi \quad (8\text{-a})$$

$$\boldsymbol{r} = \rho_0 \boldsymbol{a}_\rho \quad (8\text{-b})$$

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$$
\begin{aligned}
\boldsymbol{N}(\boldsymbol{r}) &= \iiint_{V'} \left[I_0\delta(\rho'-\rho_0)\delta(z')\boldsymbol{a}_\phi\right]e^{jk\boldsymbol{a}_r\cdot\left[\rho_0\boldsymbol{a}_{\rho'}\right]}dv' \\
&= I_0\iiint_{V'} \delta(\rho'-\rho_0)\delta(z')\left[\boldsymbol{a}_{\phi'}\right]e^{jk\rho_0\boldsymbol{a}_r\cdot\boldsymbol{a}_{\rho'}}\rho'd\rho'd\phi'dz' \\
\text{(1-a), (8)} \rightarrow \quad &= I_0\left[\int \delta(z')dz'\right]\left[\int \delta(\rho'-\rho_0)\rho'd\rho'\right]\int_0^{2\pi}\left[\boldsymbol{a}_{\phi'}\right]e^{jk\rho_0\boldsymbol{a}_r\cdot\boldsymbol{a}_{\rho'}}d\phi' \\
&= I_0\rho_0\int_0^{2\pi}\left[\boldsymbol{a}_{\phi'}\right]e^{jk\rho_0\boldsymbol{a}_r\cdot\boldsymbol{a}_{\rho'}}d\phi'
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\boldsymbol{a}_r\cdot\boldsymbol{a}_{\rho'} &= \left(\sin\theta\cos\phi\,\boldsymbol{a}_x+\sin\theta\sin\phi\,\boldsymbol{a}_y+\cos\theta\,\boldsymbol{a}_z\right)\cdot\left(\cos\phi'\,\boldsymbol{a}_x+\sin\phi'\,\boldsymbol{a}_y\right) \\
&= \sin\theta\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right) \\
&= \sin\theta\cos\Delta\phi \qquad \text{in which } \Delta\phi\equiv\phi-\phi'
\end{aligned}
\tag{TI-I}
$$

$$
\text{(9), TI-I} \rightarrow \quad \boldsymbol{N}(\boldsymbol{r}) = I_0\rho_0\int_0^{2\pi}\left[\boldsymbol{a}_{\phi'}\right]e^{jk\rho_0\sin\theta\cos\Delta\phi}d\phi' \tag{10}
$$

The N_θ and N_ϕ

$$
\begin{aligned}
N_\theta(\boldsymbol{r}) &= \boldsymbol{N}(\boldsymbol{r})\cdot\boldsymbol{a}_\theta \\
\text{(10)} \rightarrow \quad &= I_0\rho_0\int_0^{2\pi}\left[\boldsymbol{a}_{\phi'}\cdot\boldsymbol{a}_\theta\right]e^{jk\rho_0\sin\theta\cos\Delta\phi}d\phi'
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\boldsymbol{a}_{\phi'}\cdot\boldsymbol{a}_\theta &= \left(-\sin\phi'\,\boldsymbol{a}_x+\cos\phi'\,\boldsymbol{a}_y\right)\cdot\left(\cos\theta\cos\phi\,\boldsymbol{a}_x+\cos\theta\sin\phi\,\boldsymbol{a}_y\right) \\
&= \cos\theta\left(\sin\phi\cos\phi'-\cos\phi\sin\phi'\right) \\
&= \cos\theta\sin\Delta\phi \qquad \text{in which } \Delta\phi\equiv\phi-\phi'
\end{aligned}
\tag{TI-II}
$$

$$
\text{(11), TI-II} \rightarrow \quad N_\theta(\boldsymbol{r}) = I_0\rho_0\cos\theta\int_0^{2\pi}\sin\Delta\phi\,e^{jk\rho_0\sin\theta\cos\Delta\phi}d\phi' \tag{12}
$$

$$
\text{UR-III, (12)} \rightarrow \quad N_\theta(\boldsymbol{r}) = 0
$$

$$
\begin{aligned}
N_\phi(\boldsymbol{r}) &= \boldsymbol{N}(\boldsymbol{r})\cdot\boldsymbol{a}_\phi \\
\text{(10)} \rightarrow \quad &= I_0\rho_0\int_0^{2\pi}\left[\boldsymbol{a}_{\phi'}\cdot\boldsymbol{a}_\phi\right]e^{jk\rho_0\sin\theta\cos\Delta\phi}d\phi'
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\boldsymbol{a}_{\phi'}\cdot\boldsymbol{a}_\phi &= \left(-\sin\phi'\,\boldsymbol{a}_x+\cos\phi'\,\boldsymbol{a}_y\right)\cdot\left(-\sin\phi\,\boldsymbol{a}_x+\cos\phi\,\boldsymbol{a}_y\right) \\
&= \cos\phi\cos\phi'+\sin\phi\sin\phi' \\
&= \cos\Delta\phi \qquad \text{in which } \Delta\phi\equiv\phi-\phi'
\end{aligned}
\tag{TI-III}
$$

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(13), TI-III $\rightarrow$

$$N_\phi(\boldsymbol{r}) = I_0\rho_0 \int_0^{2\pi} \cos\Delta\phi\, e^{jk\rho_0 \sin\theta\cos\Delta\phi} d\phi' \tag{14}$$

UR-IV, (14) $\rightarrow$

$$\begin{aligned} N_\phi(\boldsymbol{r}) &= -j2\pi I_0\rho_0 J_0'(k\rho_0\sin\theta) \\ &= j2\pi I_0\rho_0 J_1(k\rho_0\sin\theta) \end{aligned}$$

Impulse of Magnetic Current

Magnetic Current Density

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m \delta(x)\delta(y)\delta(z)\boldsymbol{a}_z \tag{15-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{15-b}$$

(1-b), (15) $\rightarrow$

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{r}) &= \iiint_{V'} [I_0^m \delta(x')\delta(y')\delta(z')\boldsymbol{a}_z] e^{jk\boldsymbol{a}_r\cdot[z'\boldsymbol{a}_z]} dv' \\ &= I_0^m \left(\iiint_{V'} \delta(x')\delta(y')\delta(z') e^{jkz'\cos\theta} dx'dy'dz' \right) \boldsymbol{a}_z \\ &= I_0^m \left(\left[\int \delta(x')dx'\right]\left[\int \delta(y')dy'\right]\left[\int \delta(z') e^{jkz'\cos\theta} dz'\right] \right) \boldsymbol{a}_z \\ &= I_0^m \boldsymbol{a}_z \end{aligned}$$

Linear Constant Magnetic Current

Magnetic Current Density

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m \delta(x)\delta(y)\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z \tag{16-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{16-b}$$

(UR-I), (1-b), (16) $\rightarrow$

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{r}) &= \iiint_{V'} \left[I_0^m \delta(x')\delta(y')\Pi\left(\frac{z'}{l}\right)\boldsymbol{a}_z \right] e^{jk\boldsymbol{a}_r\cdot[z'\boldsymbol{a}_z]} dv' \\ &= I_0^m \left(\iiint_{V'} \delta(x')\delta(y')\Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dx'dy'dz' \right) \boldsymbol{a}_z \\ &= I_0^m \left(\left[\int \delta(x')dx'\right]\left[\int \delta(y')dy'\right]\left[\int \Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dz'\right] \right) \boldsymbol{a}_z \\ &= I_0^m \left(\int \Pi\left(\frac{z'}{l}\right) e^{jkz'\cos\theta} dz' \right) \boldsymbol{a}_z \\ &= I_0^m l \left(\int \Pi(u) e^{-j2\pi\left(-\frac{kl}{2\pi}\cos\theta\right)u} du \right) \boldsymbol{a}_z \\ &= I_0^m l \left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} \right] \boldsymbol{a}_z \end{aligned}$$

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Circular (Loop) Sinusoidal Magnetic Current

Magnetic Current Density

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m \delta(\rho - \rho_0)\cos(\phi)\,\delta(z)\boldsymbol{a}_\phi \tag{17-a}$$

$$\boldsymbol{r} = \rho_0 \boldsymbol{a}_\rho \tag{17-b}$$

$$\begin{aligned}
\boldsymbol{L}(\boldsymbol{r}) &= \iiint_{V'} \left[I_0^m \delta(\rho' - \rho_0)\cos(\phi')\,\delta(z')\boldsymbol{a}_{\phi'}\right] e^{jk\boldsymbol{a}_r \cdot \left[\rho_0 \boldsymbol{a}_{\rho'}\right]} dv' \\
&= I_0^m \iiint_{V'} \delta(\rho' - \rho_0)\cos(\phi')\,\delta(z')\left[\boldsymbol{a}_{\phi'}\right] e^{jk\rho_0 \boldsymbol{a}_r \cdot \boldsymbol{a}_{\rho'}} \rho' d\rho' d\phi' dz' \\
\text{(1-b), TI-I, (17)} \quad \rightarrow \quad &= I_0^m \left[\int \delta(z')dz'\right]\left[\int \delta(\rho' - \rho_0)\rho' d\rho'\right] \int_0^{2\pi} \left[\boldsymbol{a}_{\phi'}\right]\cos(\phi')\, e^{jk\rho_0 \boldsymbol{a}_r \cdot \boldsymbol{a}_{\rho'}} d\phi' \\
&= I_0^m \rho_0 \int_0^{2\pi} \left[\boldsymbol{a}_{\phi'}\right]\cos(\phi')\, e^{jk\rho_0 \sin\theta \cos\Delta\phi} d\phi' \qquad (18)
\end{aligned}$$

The L_θ and L_ϕ

$$\begin{aligned}
L_\theta(\boldsymbol{r}) &= \boldsymbol{L}(\boldsymbol{r}) \cdot \boldsymbol{a}_\theta \\
&= I_0^m \rho_0 \int_0^{2\pi} \left[\boldsymbol{a}_{\phi'} \cdot \boldsymbol{a}_\theta\right]\cos(\phi')\, e^{jk\rho_0 \sin\theta \cos\Delta\phi} d\phi' \\
\text{UR-III, (1-b), TI-II, (18)} \quad \rightarrow \quad &= I_0^m \rho_0 \cos\theta \int_0^{2\pi} \sin\Delta\phi \cos(\phi')\, e^{jk\rho_0 \sin\theta \cos\Delta\phi} d\phi' \\
&= -I_0^m (2\pi\rho_0)\cos\theta \sin\phi \left[\frac{J_1(k\rho_0 \sin\theta)}{k\rho_0 \sin\theta}\right]
\end{aligned}$$

$$\begin{aligned}
L_\phi(\boldsymbol{r}) &= \boldsymbol{L}(\boldsymbol{r}) \cdot \boldsymbol{a}_\phi \\
&= I_0^m \rho_0 \int_0^{2\pi} \left[\boldsymbol{a}_{\phi'} \cdot \boldsymbol{a}_\phi\right]\cos(\phi')\, e^{jk\rho_0 \sin\theta \cos\Delta\phi} d\phi' \\
\text{UR-IV, TI-III, (18)} \quad \rightarrow \quad &= I_0^m \rho_0 \int_0^{2\pi} \cos\Delta\phi \cos(\phi')\, e^{jk\rho_0 \sin\theta \cos\Delta\phi} d\phi' \\
&= I_0^m (2\pi\rho_0)\cos\phi\, J_1'(k\rho_0 \sin\theta)
\end{aligned}$$

Uniformly Illuminated Rectangular Aperture

Illumination

$$\boldsymbol{E}(\boldsymbol{r}) = E_0 \Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\delta(z)\boldsymbol{a}_y \tag{19}$$

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Magnetic Current Density

Magnetic current density is derived from the electric field that illuminates the aperture using the Surface Equivalence Theorem. (20)

(19), (20) →

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{r}) &= -2\boldsymbol{n}\times\boldsymbol{E}(\boldsymbol{r}) \\ &= I_0^m \Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\delta(z)\boldsymbol{a}_x \qquad \text{in which } I_0^m \equiv 2E_0 \end{aligned} \tag{21-a}$$

$$\boldsymbol{r} = x\boldsymbol{a}_x + y\boldsymbol{a}_y \tag{21-b}$$

(UR-I), (1-b), (21) →

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{r}) &= \iiint_{V'}\left[I_0^m \Pi\left(\frac{x'}{a}\right)\Pi\left(\frac{y'}{b}\right)\delta(z')\boldsymbol{a}_x\right] e^{jk\boldsymbol{a}_r\cdot[x'\boldsymbol{a}_x+y'\boldsymbol{a}_y]}dv' \\ &= I_0^m\left(\iiint_{V'}\Pi\left(\frac{x'}{a}\right)\Pi\left(\frac{y'}{b}\right)\delta(z')e^{jk\boldsymbol{a}_r\cdot[x'\boldsymbol{a}_x+y'\boldsymbol{a}_y]}dx'dy'dz'\right)\boldsymbol{a}_x \\ &= I_0^m\left(\left[\int\delta(z')dz'\right]\left[\int\Pi\left(\frac{x'}{a}\right)e^{jkx'\boldsymbol{a}_r\cdot\boldsymbol{a}_x}dx'\right]\left[\int\Pi\left(\frac{y'}{b}\right)e^{jky'\boldsymbol{a}_r\cdot\boldsymbol{a}_y}dy'\right]\right)\boldsymbol{a}_x \\ &= I_0^m\left(\left[\int\Pi\left(\frac{x'}{a}\right)e^{jkx'\sin\theta\cos\phi}dx'\right]\left[\int\Pi\left(\frac{y'}{b}\right)e^{jky'\sin\theta\sin\phi}dy'\right]\right)\boldsymbol{a}_x \\ &= I_0^m ab\left(\left[\int\Pi(u)e^{-j2\pi\left(-\frac{ka}{2\pi}\sin\theta\cos\phi\right)u}du\right]\left[\int\Pi(u)e^{-j2\pi\left(-\frac{kb}{2\pi}\sin\theta\sin\phi\right)u}du\right]\right)\boldsymbol{a}_x \\ &= I_0^m ab\left[\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}\sin\theta\cos\phi}\right]\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\boldsymbol{a}_x \end{aligned}$$

Uniformly Illuminated Circular Aperture

Illumination

$$\boldsymbol{E}(\boldsymbol{r}) = E_0\delta(z)\boldsymbol{a}_y \tag{22-a}$$

$$\boldsymbol{r} = \rho\boldsymbol{a}_\rho \text{ and } \rho \le \rho_0 \tag{22-b}$$

Magnetic Current Density

Magnetic current density is derived from the electric field that illuminates the aperture using the Surface Equivalence Theorem. (23)

(22), (23) →

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{r}) &= -2\boldsymbol{n}\times\boldsymbol{E}(\boldsymbol{r}) \\ &= I_0^m\delta(z)\boldsymbol{a}_x \qquad \text{in which } I_0^m \equiv 2E_0 \end{aligned} \tag{24-a}$$

$$\boldsymbol{r} = \rho\boldsymbol{a}_\rho \text{ and } \rho \le \rho_0 \tag{24-b}$$

(1-b), (24) →

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{r}) &= \iiint_{V'}[I_0^m\delta(z)\boldsymbol{a}_x]e^{jk\boldsymbol{a}_r\cdot\left[\rho'\boldsymbol{a}_{\rho'}\right]}dv' \\ &= I_0^m\left(\iiint_{V'}\delta(z')e^{jk\rho'\boldsymbol{a}_r\cdot\boldsymbol{a}_{\rho'}}\rho'd\rho'd\phi'dz'\right)\boldsymbol{a}_x \end{aligned} \tag{25}$$

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$$= I_0^m \left(\left[\int \delta(z')dz' \right] \left[\int_0^{\rho_0} \int_0^{2\pi} e^{jk\rho' \boldsymbol{a}_r \cdot \boldsymbol{a}_{\rho'}} d\phi' \rho' d\rho' \right] \right) \boldsymbol{a}_x$$

$$= I_0^m \left(\int_0^{\rho_0} \left[\int_0^{2\pi} e^{jk\rho' \boldsymbol{a}_r \cdot \boldsymbol{a}_{\rho'}} d\phi' \right] \rho' d\rho' \right) \boldsymbol{a}_x$$

$$\begin{aligned} \boldsymbol{a}_r \cdot \boldsymbol{a}_{\rho'} &= (\sin\theta\cos\phi\,\boldsymbol{a}_x + \sin\theta\sin\phi\,\boldsymbol{a}_y + \cos\theta\,\boldsymbol{a}_z) \cdot (\cos\phi'\,\boldsymbol{a}_x + \sin\phi'\,\boldsymbol{a}_y) \\ &= \sin\theta\,(\cos\phi\cos\phi' + \sin\phi\sin\phi') \\ &= \sin\theta\cos\Delta\phi \qquad \text{in which } \Delta\phi \equiv \phi - \phi' \end{aligned} \tag{TI-IV}$$

$$\boldsymbol{L}(\boldsymbol{r}) = I_0^m \left(\int_0^{\rho_0} \left[\int_0^{2\pi} e^{jk\rho' \sin\theta\cos\Delta\phi} d\phi' \right] \rho' d\rho' \right) \boldsymbol{a}_x$$

$$= 2\pi I_0^m \left(\int_0^{\rho_0} \rho' J_0(k\rho' \sin\theta) d\rho' \right) \boldsymbol{a}_x$$

UR-V, UR-VI, (25), (TI-IV) →

$$= \frac{2\pi}{(k\sin\theta)^2} I_0^m \left(\int_0^{k\rho_0 \sin\theta} [k\rho' \sin\theta] J_0(k\rho' \sin\theta) [k\sin\theta\, d\rho'] \right) \boldsymbol{a}_x$$

$$= \frac{2\pi}{(k\sin\theta)^2} I_0^m \left(\int_0^{k\rho_0 \sin\theta} u J_0(u) du \right) \boldsymbol{a}_x$$

$$= 2(\pi\rho_0^2) I_0^m \left[\frac{J_1(k\rho_0 \sin\theta)}{k\rho_0 \sin\theta} \right] \boldsymbol{a}_x$$

Table 16. Derivation of the Radiation Pattern of Two-Element Arrays of Impulses

Radiated Fields in the Far-Field Region

$$E_\theta = -jk\eta \frac{e^{-jkr}}{4\pi r} \left(N_\theta + \frac{1}{\eta} L_\phi \right) \tag{1-a}$$

$$E_\phi = -jk\eta \frac{e^{-jkr}}{4\pi r} \left(N_\phi - \frac{1}{\eta} L_\theta \right) \tag{1-b}$$

$$\boldsymbol{H} = \frac{1}{\eta} \boldsymbol{a}_r \times \boldsymbol{E}$$

Radiation Pattern of an Impulse

Impulse of Electric Current in the Direction of $\boldsymbol{a}_z$

$$N_\theta(\boldsymbol{r}) = -I_0 \sin\theta \tag{2-a}$$

$$N_\phi(\boldsymbol{r}) = 0 \tag{2-b}$$

$$\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{0} \tag{2-c}$$

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(1), (2) → $$\boldsymbol{E} = jk\eta I_0 \frac{e^{-jkr}}{4\pi r} \sin\theta\, \boldsymbol{a}_\theta \quad (3)$$

Impulse of Electric Current in the Direction of $\boldsymbol{a}_y$

$$N_\theta(\boldsymbol{r}) = I_0 \cos\theta \sin\phi \quad (4\text{-a})$$
$$N_\phi(\boldsymbol{r}) = I_0 \cos\phi \quad (4\text{-b})$$
$$\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{0} \quad (4\text{-c})$$

(1), (4) → $$\boldsymbol{E} = -jk\eta I_0 \frac{e^{-jkr}}{4\pi r} (\cos\theta \sin\phi\, \boldsymbol{a}_\theta + \cos\phi\, \boldsymbol{a}_\phi) \quad (5)$$

Impulse of Magnetic Current in the Direction of $\boldsymbol{a}_z$

$$\boldsymbol{N}(\boldsymbol{r}) = \boldsymbol{0} \quad (6\text{-a})$$
$$L_\theta(\boldsymbol{r}) = -I_0^m \sin\theta \quad (6\text{-b})$$
$$L_\phi(\boldsymbol{r}) = 0 \quad (6\text{-c})$$

(1), (6) → $$\boldsymbol{E} = -jkI_0^m \frac{e^{-jkr}}{4\pi r} \sin\theta\, \boldsymbol{a}_\phi \quad (7)$$

Impulse of Magnetic Current in the Direction of $\boldsymbol{a}_y$

$$\boldsymbol{N}(\boldsymbol{r}) = \boldsymbol{0} \quad (8\text{-a})$$
$$L_\theta(\boldsymbol{r}) = I_0^m \cos\theta \sin\phi \quad (8\text{-b})$$
$$L_\phi(\boldsymbol{r}) = I_0^m \cos\phi \quad (8\text{-c})$$

(1), (8) → $$\boldsymbol{E} = jkI_0^m \frac{e^{-jkr}}{4\pi r} \left(\cos\phi\, \boldsymbol{a}_\theta + \cos\theta \sin\phi\, \boldsymbol{a}_\phi\right) \quad (9)$$

Two-Element Array Factor

$$AF_N = \sum_{n=1}^{N} A_n e^{jku_n \boldsymbol{a}_r \cdot \boldsymbol{a}_u} \quad (10)$$

Array elements are located along the z axis. (11)

(10), (11) → $$AF_N = \sum_{n=1}^{N} A_n e^{jkz_n \cos\theta} \quad (12)$$

In-Phase Elements

For first element, $A_1 = 1$, $z_1 = -d$. For the second element, $A_2 = 1$, $z_2 = d$. (13)

(12), (13) → $$AF_2 = 2\cos(kd\cos\theta) \quad (14)$$

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Anti-Phase Elements

For first element, $A_1 = -1$, $z_1 = -d$. For the second element, $A_2 = 1$, $z_2 = d$. (15)

(12), (15) → $$AF_2 = j2\sin(kd\cos\theta) \quad (16)$$

Radiation Pattern for Two-Element Arrays of Impulses

In-Phase Array of $\boldsymbol{a}_z$- Directed Electric Current Impulses

(3), (14) → $$\boldsymbol{E} = j2k\eta I_0 \frac{e^{-jkr}}{4\pi r}\cos(kd\cos\theta)\sin\theta\,\boldsymbol{a}_\theta$$

This configuration is equivalent to an $\boldsymbol{a}_z$-directed impulse of electric current above a PEC (ground) plane.

Anti-Phase Array of $\boldsymbol{a}_z$- Directed Electric Current Impulses

(3), (16) → $$\boldsymbol{E} = -2k\eta I_0 \frac{e^{-jkr}}{4\pi r}\sin(kd\cos\theta)\sin\theta\,\boldsymbol{a}_\theta$$

In-Phase Array of $\boldsymbol{a}_y$- Directed Electric Current Impulses

(5), (14) → $$\boldsymbol{E} = -j2k\eta I_0 \frac{e^{-jkr}}{4\pi r}\cos(kd\cos\theta)\left(\cos\theta\sin\phi\,\boldsymbol{a}_\theta + \cos\phi\,\boldsymbol{a}_\phi\right)$$

Anti-Phase Array of $\boldsymbol{a}_y$- Directed Electric Current Impulses

(5), (16) → $$\boldsymbol{E} = 2k\eta I_0 \frac{e^{-jkr}}{4\pi r}\sin(kd\cos\theta)\left(\cos\theta\sin\phi\,\boldsymbol{a}_\theta + \cos\phi\,\boldsymbol{a}_\phi\right)$$

This configuration is equivalent to an $\boldsymbol{a}_y$-directed impulse of electric current above a PEC (ground) plane.

In-Phase Array of $\boldsymbol{a}_z$- Directed Magnetic Current Impulses

(7), (14) → $$\boldsymbol{E} = -j2kI_0^m \frac{e^{-jkr}}{4\pi r}\cos(kd\cos\theta)\sin\theta\,\boldsymbol{a}_\phi$$

Anti-Phase Array of $\boldsymbol{a}_z$- Directed Magnetic Current Impulses

(7), (16) → $$\boldsymbol{E} = 2kI_0^m \frac{e^{-jkr}}{4\pi r}\sin(kd\cos\theta)\sin\theta\,\boldsymbol{a}_\phi$$

This configuration is equivalent to an $\boldsymbol{a}_z$-directed impulse of magnetic current above a PEC (ground) plane.

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In-Phase Array of $\boldsymbol{a}_y$- Directed Magnetic Current Impulses

(9), (14) → $$\boldsymbol{E} = j2kI_0^m \frac{e^{-jkr}}{4\pi r} \cos(kd\cos\theta)\,(\cos\phi\,\boldsymbol{a}_\theta + \cos\theta\sin\phi\,\boldsymbol{a}_\phi)$$

This configuration is equivalent to an $\boldsymbol{a}_y$-directed impulse of magnetic current above a PEC (ground) plane.

Anti-Phase Array of $\boldsymbol{a}_y$- Directed Magnetic Current Impulses

(9), (16) → $$\boldsymbol{E} = -2kI_0^m \frac{e^{-jkr}}{4\pi r} \sin(kd\cos\theta)\,(\cos\phi\,\boldsymbol{a}_\theta + \cos\theta\sin\phi\,\boldsymbol{a}_\phi)$$

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6- Propagation in Material Media

Table 17. Reformulation of the Uniform Plane Wave in Lossy Dielectric Media

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \quad (1\text{-a})$$

$$\boldsymbol{H} = \frac{1}{\omega\mu}\boldsymbol{k} \times \boldsymbol{E} \quad (1\text{-b})$$

$$\boldsymbol{E}_0 = E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y + E_{0z}\boldsymbol{a}_z \quad (1\text{-c})$$

$$\boldsymbol{k} = k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y + k_z\boldsymbol{a}_z \quad (1\text{-d})$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon_c \quad (1\text{-e})$$

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \quad (1\text{-f})$$

Defining $k_x \equiv -j\gamma\sin\theta\cos\phi$, $k_y \equiv -j\gamma\sin\theta\sin\phi$ and $k_z \equiv -j\gamma\cos\theta$, then, from (1-d), $\boldsymbol{k}$ can be rewritten as $\boldsymbol{k} = -j\gamma\boldsymbol{a}_{r,p}$ in which $\boldsymbol{a}_{r,p}$ is the unit vector associated with the spherical coordinate r that is pointing in the direction of propagation. (2)

(1-a), (2) → $\boldsymbol{E} = \boldsymbol{E}_0 e^{-\gamma\boldsymbol{a}_{r,p}\cdot\boldsymbol{r}}$

(1-b), (2) → $\boldsymbol{H} = \frac{1}{\eta_c}\boldsymbol{a}_{r,p} \times \boldsymbol{E}$ in which $\eta_c \equiv j\frac{\omega\mu}{\gamma}$

(1-e), (2) → $\gamma^2 = -\omega^2\mu\varepsilon_c$

(1-f), (2) → $\boldsymbol{a}_{r,p}\cdot\boldsymbol{E}_0 = 0$

Table 18. Derivation of Group Velocity

The group velocity is defined as the speed of the wave envelope. (1)

The Envelope of the Sum Wave

Assuming that two waves with equal amplitudes but different frequencies f_1 and f_2 are propagating in a lossless medium, then the time-harmonic sum wave is given by $E(t) = Re\left(E_0 e^{-j(\beta_1 z-\omega_1 t)} + E_0 e^{-j(\beta_2 z-\omega_2 t)}\right)$. By defining $\theta_1 \equiv \beta_1 z - \omega_1 t$ and $\theta_2 \equiv \beta_2 z - \omega_2 t$, the expression for the time-harmonic sum wave can be simplified to $E(t) = Re\left(E_0 e^{-j\theta_1} + E_0 e^{-j\theta_2}\right)$. (2)

By defining $\theta^{\Sigma} \equiv \theta_1 + \theta_2$ and $\theta^{\Delta} \equiv \theta_1 - \theta_2$, the θ_1 and θ_2 can be rewritten as $\theta_1 = \frac{1}{2}\left(\theta^{\Sigma}\right) + \frac{1}{2}\left(\theta^{\Delta}\right)$ and $\theta_2 = \frac{1}{2}\left(\theta^{\Sigma}\right) - \frac{1}{2}\left(\theta^{\Delta}\right)$. (3)

(2), (3) → $E(t) = Re\left(\left[E_0 e^{-j\left(\frac{1}{2}\theta^{\Delta}\right)} + E_0 e^{j\left(\frac{1}{2}\theta^{\Delta}\right)}\right] e^{-j\left(\frac{\beta_1+\beta_2}{2}\right)z} e^{j\left(\frac{\omega_1+\omega_2}{2}\right)t}\right)$ (4)

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(4) → The envelope is given by $E_0 \cos\left(\frac{1}{2}\theta^{\Delta}\right)$. (5)

(1), (5) → $\frac{\partial}{\partial t}\left[E_0 \cos\left(\frac{1}{2}\theta^{\Delta}\right)\right] = 0$ (6)

(6) → $\Delta\beta \frac{\partial z}{\partial t} - \Delta\omega = 0$ (7)

(7) → $u_g \equiv \frac{\partial z}{\partial t} = \frac{\Delta\omega}{\Delta\beta}$

Table 19. Derivation of the Reflection and Transmission Coefficients at the Interface of Media

Dielectric-Dielectric Interface

The Parallel (Vertical/H) Mode

The Incident Field

$$\boldsymbol{a}_{r,p}^{i} = \sin\theta_i\, \boldsymbol{a}_x + \cos\theta_i\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{i} = E_0(\cos\theta_i\, \boldsymbol{a}_x - \sin\theta_i\, \boldsymbol{a}_z)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{i} = \frac{1}{\eta_1}(E_0\boldsymbol{a}_y)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}} \quad (1)$$

The Reflected Field

$$\boldsymbol{a}_{r,p}^{r} = \sin\theta_r\, \boldsymbol{a}_x - \cos\theta_r\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{r} = \Gamma E_0(\cos\theta_r\, \boldsymbol{a}_x + \sin\theta_r\, \boldsymbol{a}_z)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{r} = \frac{1}{\eta_1}\Gamma(-E_0\boldsymbol{a}_y)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}} \quad (2)$$

The Transmitted Field

$$\boldsymbol{a}_{r,p}^{t} = \sin\theta_t\, \boldsymbol{a}_x + \cos\theta_t\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{t} = \mathrm{T}E_0(\cos\theta_t\, \boldsymbol{a}_x - \sin\theta_t\, \boldsymbol{a}_z)e^{-\gamma_2 \boldsymbol{a}_{r,p}^{t}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{t} = \frac{1}{\eta_2}\mathrm{T}(E_0\boldsymbol{a}_y)e^{-\gamma_2 \boldsymbol{a}_{r,p}^{t}\cdot\boldsymbol{r}} \quad (3)$$

The Boundary Conditions at the Interface

Along the interface, which is described by $\boldsymbol{r} = x\boldsymbol{a}_x$, the boundary conditions are given by $\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$ and $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0}$ in which $\boldsymbol{n} = \boldsymbol{a}_z$. (4)

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(1), (2), (3), (4) →

$$\cos\theta_i\, e^{-\gamma_1 \sin\theta_i x} + \Gamma \cos\theta_r\, e^{-\gamma_1 \sin\theta_r x} = T \cos\theta_t\, e^{-\gamma_2 \sin\theta_t x} \quad (5\text{-a})$$

$$\frac{1}{\eta_1} e^{-\gamma_1 \sin\theta_i x} - \frac{1}{\eta_1}\Gamma e^{-\gamma_1 \sin\theta_r x} = \frac{1}{\eta_2}\mathrm{T} e^{-\gamma_2 \sin\theta_t x} \quad (5\text{-b})$$

Along the interface, the phase terms must be equal, i.e., $\gamma_1 \sin\theta_i\, x = \gamma_1 \sin\theta_r\, x = \gamma_2 \sin\theta_t\, x$. Therefore, $\theta_r = \theta_i$ and $\sin\theta_t = \frac{\gamma_1}{\gamma_2}\sin\theta_i$. (6)

(5), (6) →

$$T = \frac{\cos\theta_i}{\cos\theta_t}(1+\Gamma)$$
$$T = \frac{\eta_2}{\eta_1}(1-\Gamma)$$

The Perpendicular (Horizontal/E) Mode

The Incident Field

$$\boldsymbol{a}^i_{r,p} = \sin\theta_i\, \boldsymbol{a}_x + \cos\theta_i\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^i = \left(E_0 \boldsymbol{a}_y\right) e^{-\gamma_1 \boldsymbol{a}^i_{r,p}\cdot \boldsymbol{r}}$$
$$\boldsymbol{H}^i = \frac{1}{\eta_1} E_0 (-\cos\theta_i\, \boldsymbol{a}_x + \sin\theta_i\, \boldsymbol{a}_z) e^{-\gamma_1 \boldsymbol{a}^i_{r,p}\cdot \boldsymbol{r}} \quad (7)$$

The Reflected Field

$$\boldsymbol{a}^r_{r,p} = \sin\theta_r\, \boldsymbol{a}_x - \cos\theta_r\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^r = \Gamma\left(E_0 \boldsymbol{a}_y\right) e^{-\gamma_1 \boldsymbol{a}^r_{r,p}\cdot \boldsymbol{r}}$$
$$\boldsymbol{H}^r = \frac{1}{\eta_1}\Gamma E_0 (\cos\theta_r\, \boldsymbol{a}_x + \sin\theta_r\, \boldsymbol{a}_z) e^{-\gamma_1 \boldsymbol{a}^r_{r,p}\cdot \boldsymbol{r}} \quad (8)$$

The Transmitted Field

$$\boldsymbol{a}^t_{r,p} = \sin\theta_t\, \boldsymbol{a}_x + \cos\theta_t\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^t = \mathrm{T}\left(E_0 \boldsymbol{a}_y\right) e^{-\gamma_2 \boldsymbol{a}^t_{r,p}\cdot \boldsymbol{r}}$$
$$\boldsymbol{H}^t = \frac{1}{\eta_2}\mathrm{T} E_0 (-\cos\theta_t\, \boldsymbol{a}_x + \sin\theta_t\, \boldsymbol{a}_z) e^{-\gamma_2 \boldsymbol{a}^t_{r,p}\cdot \boldsymbol{r}} \quad (9)$$

The Boundary Conditions at the Interface

Along the interface, which is described by $\boldsymbol{r} = x\boldsymbol{a}_x$, the boundary conditions are given by $\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$ and $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0}$ in which $\boldsymbol{n} = \boldsymbol{a}_z$. (10)

(7), (8), (9), (10) →

$$e^{-\gamma_1 \sin\theta_i x} + \Gamma e^{-\gamma_1 \sin\theta_r x} = T e^{-\gamma_2 \sin\theta_t x} \quad (11\text{-a})$$

$$\frac{1}{\eta_1}\cos\theta_i\, e^{-\gamma_1 \sin\theta_i x} + \frac{1}{\eta_1}\Gamma \cos\theta_r\, e^{-\gamma_1 \sin\theta_r x} = \frac{1}{\eta_2} T \cos\theta_t\, e^{-\gamma_2 \sin\theta_t x} \quad (11\text{-b})$$

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Along the interface, the phase terms must be equal, i.e., $\gamma_1 \sin\theta_i\, x = \gamma_1 \sin\theta_r\, x = \gamma_2 \sin\theta_t\, x$. Therefore, $\theta_r = \theta_i$ and $\sin\theta_t = \frac{\gamma_1}{\gamma_2}\sin\theta_i$. (12)

(11), (12) → $T = 1 + \Gamma$

$$T = \frac{\cos\theta_i}{\cos\theta_t}\frac{\eta_2}{\eta_1}(1 - \Gamma)$$

Dielectric-PEC Interface

The Parallel (Vertical/H) Mode

The Incident Field

$$\boldsymbol{a}_{r,p}^{i} = \sin\theta_i\, \boldsymbol{a}_x + \cos\theta_i\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{i} = E_0(\cos\theta_i\, \boldsymbol{a}_x - \sin\theta_i\, \boldsymbol{a}_z)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{i} = \frac{1}{\eta_1}\left(E_0 \boldsymbol{a}_y\right)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}} \quad (13)$$

The Reflected Field

$$\boldsymbol{a}_{r,p}^{r} = \sin\theta_r\, \boldsymbol{a}_x - \cos\theta_r\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{r} = \Gamma E_0(\cos\theta_r\, \boldsymbol{a}_x + \sin\theta_r\, \boldsymbol{a}_z)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{r} = \frac{1}{\eta_1}\Gamma\left(-E_0 \boldsymbol{a}_y\right)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}} \quad (14)$$

The Boundary Condition at the Interface

Since the second medium is PEC, $\boldsymbol{E}_2 = \boldsymbol{0}$ and $\boldsymbol{H}_2 = \boldsymbol{0}$. Along the interface, which is described by $\boldsymbol{r} = x\boldsymbol{a}_x$, the boundary conditions are given by $\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$ and $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$ in which $\boldsymbol{n} = \boldsymbol{a}_z$. (15)

(13), (14), (15) → $\cos\theta_i\, e^{-\gamma_1 \sin\theta_i x} + \Gamma \cos\theta_r\, e^{-\gamma_1 \sin\theta_r x} = 0$ (16)

Along the interface, the phase terms must be equal, i.e., $\gamma_1 \sin\theta_i\, x = \gamma_1 \sin\theta_r\, x$. Therefore, $\theta_r = \theta_i$. (17)

(16), (17) → $\Gamma = -1$

The Perpendicular (Horizontal/E) Mode

The Incident Field

$$\boldsymbol{a}_{r,p}^{i} = \sin\theta_i\, \boldsymbol{a}_x + \cos\theta_i\, \boldsymbol{a}_z$$
$$\boldsymbol{E}^{i} = \left(E_0 \boldsymbol{a}_y\right)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^{i} = \frac{1}{\eta_1}E_0(-\cos\theta_i\, \boldsymbol{a}_x + \sin\theta_i\, \boldsymbol{a}_z)e^{-\gamma_1 \boldsymbol{a}_{r,p}^{i}\cdot\boldsymbol{r}} \quad (18)$$

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The Reflected Field

$$\boldsymbol{a}_{r,p}^{r} = \sin\theta_r\,\boldsymbol{a}_x - \cos\theta_r\,\boldsymbol{a}_z$$
$$\boldsymbol{E}^r = \Gamma\big(E_0\boldsymbol{a}_y\big)e^{-\gamma_1\boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^r = \frac{1}{\eta_1}\Gamma E_0(\cos\theta_r\,\boldsymbol{a}_x + \sin\theta_r\,\boldsymbol{a}_z)e^{-\gamma_1\boldsymbol{a}_{r,p}^{r}\cdot\boldsymbol{r}} \quad (19)$$

The Boundary Condition at the Interface

Since the second medium is PEC, $\boldsymbol{E}_2 = \boldsymbol{0}$ and $\boldsymbol{H}_2 = \boldsymbol{0}$. Along the interface, which is described by $\boldsymbol{r} = x\boldsymbol{a}_x$, the boundary conditions are given by $\boldsymbol{n}\times(\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$ and $\boldsymbol{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$ in which $\boldsymbol{n} = \boldsymbol{a}_z$. (20)

(18), (19), (20) → $e^{-\gamma_1\sin\theta_i x} + \Gamma e^{-\gamma_1\sin\theta_r x} = 0$ (21)

Along the interface, the phase terms must be equal, i.e., $\gamma_1\sin\theta_i\,x = \gamma_1\sin\theta_r\,x$. Therefore, $\theta_r = \theta_i$. (22)

(21), (22) → $\Gamma = -1$

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Part III

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7- Antenna Elements and Arrays

Table 20. Derivation of the Radiation Pattern of Antenna Elements

Radiated Fields in the Far-Field Region

$$E_r = 0$$

$$E_\theta = -jk\eta \frac{e^{-jkr}}{4\pi r}\left(N_\theta + \frac{1}{\eta}L_\phi\right) \quad \text{(1-a)}$$

$$E_\phi = -jk\eta \frac{e^{-jkr}}{4\pi r}\left(N_\phi - \frac{1}{\eta}L_\theta\right) \quad \text{(1-b)}$$

$$\boldsymbol{H} = \frac{1}{\eta}\boldsymbol{a}_r \times \boldsymbol{E}$$

Useful Relations

Bessel Function Small Argument Approximation

For small values of x, i.e., when $x \to 0$, $J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n$ UR-I

Infinitesimal Dipole

Electric Current Density

$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z$ and $\frac{l}{\lambda} \ll 1$

Radiation Pattern of the Linear Constant Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\Pi\left(\frac{z}{l}\right)\boldsymbol{a}_z$$

Far-Field Radiation Integral

$$N_\theta = -(lI_0)\left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta}\right]\sin\theta \quad \text{(2-a)}$$

$$N_\phi = 0 \quad \text{(2-b)}$$

Imposing the condition $\frac{l}{\lambda} \ll 1$ on (2-a) results in $\left[\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta}\right] \approx 1$ (3)

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(2), (3) → $N_\theta = -(lI_0)\ \sin\theta$ (4-a)

$N_\phi = 0$ (4-b)

(1), (4) →

$$E_r = 0$$

$$E_\theta = jk\eta(lI_0)\frac{e^{-jkr}}{4\pi r}\sin\theta$$

$$E_\phi = 0$$

Half-Wave (Resonant) Dipole

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\sin\left(k\left(\frac{l}{2} - |z|\right)\right)\boldsymbol{a}_z \text{ in which } |z| \le \frac{l}{2} \text{ and } l = \frac{\lambda}{2}$$

Radiation Pattern of the Linear Sinusoidal Electric Current Density

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(x)\delta(y)\sin\left(k\left(\frac{l}{2} - |z|\right)\right)\boldsymbol{a}_z \text{ in which } |z| \le \frac{l}{2}$$

Far-Field Radiation Integral

$$N_\theta = -\frac{2}{k}I_0\left(\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right) \quad (5\text{-a})$$

$$N_\phi = 0 \quad (5\text{-b})$$

The length of the distribution is set to $l = \frac{\lambda}{2}$ (6)

(5), (6) →

$$N_\theta = -\frac{2}{k}I_0\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right) \quad (7\text{-a})$$

$$N_\phi = 0 \quad (7\text{-b})$$

(1), (7) →

$$E_r = 0$$

$$E_\theta = j2\eta I_0\frac{e^{-jkr}}{4\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$E_\phi = 0$$

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Small Loop

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\delta(z)\boldsymbol{a}_\phi \text{ and } \frac{2\pi\rho_0}{\lambda} \ll 1$$

Radiation Pattern of the Circular Constant Electric Current

Electric Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\delta(z)\boldsymbol{a}_\phi$$

Far-Field Radiation Integral

$$N_\theta = 0 \quad (8\text{-a})$$
$$N_\phi = j([2\pi\rho_0]I_0)J_1(k\rho_0\sin\theta) \quad (8\text{-b})$$

UR-I → Imposing the condition $\frac{2\pi\rho_0}{\lambda} \ll 1$ on (8-b) results in $J_1(k\rho_0\sin\theta) \approx \frac{1}{2}k\rho_0\sin\theta$ (9)

(8), (9) →
$$N_\theta = 0 \quad (10\text{-a})$$
$$N_\phi = jk(\pi\rho_0^2)I_0\sin\theta \quad (10\text{-b})$$

(1), (10) →
$$E_r = 0$$
$$E_\theta = 0$$
$$E_\phi = \frac{\eta}{4\pi}k^2\rho_0^2 I_0^2\frac{e^{-jkr}}{r}\sin\theta$$

Rectangular Microstrip Patch

The rectangular patch is located at $z = \frac{h}{2}$ and extends from $x = -\frac{a}{2}$ to $x = \frac{a}{2}$ along the x axis and from $y = -\frac{b}{2}$ to $y = \frac{b}{2}$ along the y axis. The ground plane is located at $z = -\frac{h}{2}$. The space between the patch and the ground plane (height h) is filled with dielectric material. The perimeter of the patch (which is parallel to the $x-y$ plane) and the thickness of the dielectric material along the z axis define a surface which, here, is referred to as surface S. (11-a)

The height h is assumed to be infinitesimally small, i.e., $\frac{h}{\lambda} \ll 1$. This implies that there are no variations in the fields along the z axis, i.e., $\frac{\partial}{\partial z} = 0$. Therefore, the fields underneath the patch (i.e., within the thickness of the dielectric), which are denoted by $\boldsymbol{E}^{patch}$ and $\boldsymbol{H}^{patch}$, can be described using 2D modes. (11-b)

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The electric current density $\boldsymbol{J}$ over the surface of the patch represents the discontinuity in the magnetic fields on both sides of the patch. While $\boldsymbol{H}^{patch}$ is non-zero, the magnetic field above the patch is zero. Therefore, $\boldsymbol{J} = -\boldsymbol{a}_z \times \boldsymbol{H}^{patch}$. (11-c)

The boundary condition imposed on the surface S must ensure that the electric current density over the patch does not have a normal component to the perimeter of the patch at the perimeter itself. In other words, $\boldsymbol{n}_b \cdot \boldsymbol{J} = 0$ in which $\boldsymbol{n}_b$ represents a unit vector normal to the perimeter at any point along the perimeter.

(11-b), (11-c) → From the above condition and (11-c) it follows that $H_x^{patch} = 0$ along the edges that are parallel to the x axis and $H_y^{patch} = 0$ along the edges that are parallel to the y axis. This means that along the entire perimeter of the patch the tangential component of the magnetic field is zero. In other words, the surface S is a Perfect Magnetic Conductor (PMC). (11-d)

(11-b), (11-c) → The modes that can exist underneath the patch can be determined by considering the components of the electric current density $\boldsymbol{J}$. From (11-c) it follows that $J_x = H_y^{patch}$ and that $J_y = -H_x^{patch}$. Therefore, for $\boldsymbol{J}$ to not wannish, $\boldsymbol{H}^{patch}$ must have at least one component in the direction of $\boldsymbol{a}_x$ or $\boldsymbol{a}_y$. The 2D TE^z mode which only has a magnetic field in the direction of $\boldsymbol{a}_z$ can't satisfy the above condition. Therefore, the only modes that can exist underneath the patch are 2D TM^z. (11-e)

The 2D TM^z Mode

To simplify calculations, in this section, the rectangular patch is assumed to be located at $z = h$ and to extend from $x = 0$ to $x = a$ along the x axis and from $y = 0$ to $y = b$ along the y axis. This is a simple shift of the coordinate system relative to (11-a) and does not affect the physics of the problem.

$$\begin{aligned} E_x &= 0 & H_x &= \frac{1}{\mu}\frac{\partial A_z}{\partial y} \\ E_y &= 0 & H_y &= -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \\ E_z &= -j\omega A_z & H_z &= 0 \end{aligned} \quad (12\text{-a})$$

In the above equations, A_z satisfies the following PDE:

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \omega^2 \mu\varepsilon A_z = 0 \quad (12\text{-b})$$

(11-d), (12-b) → $$A_z^{mn} = A_0 \cos\left(m\frac{x}{a}\pi\right)\cos\left(n\frac{y}{b}\pi\right) \text{ in which } \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \omega^2\mu\varepsilon \quad (12\text{-c})$$

The Electric Field for the Dominant Mode TM_{10}^z

(12-a), (12-c) → $$E_z^{TM_{10}^z} = -j\omega A_0 \cos\left(\frac{x}{a}\pi\right) \text{ in which } f_r = \frac{1}{2a\sqrt{\mu\varepsilon}} \quad (13)$$

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Magnetic Current Density

To simplify calculations, in this section, the rectangular patch is, initially, assumed to be located at $z = h$ and to extend from $x = 0$ to $x = a$ along the x axis and from $y = 0$ to $y = b$ along the y axis. This is a simple shift of the coordinate system relative to (11-a) and does not affect the physics of the problem. Once the distribution of the magnetic current density over apertures are determined, derivations are continued using the assumptions of (11-a).

Surface S, as defined in (11-a), comprises four apertures. The magnetic current density $\boldsymbol{M}^{TM_{10}^z}$ over each aperture represents the discontinuity in the electric fields on both sides of that aperture. While $\boldsymbol{E}^{TM_{10}^z}$ is non-zero, the electric field beyond the patch is zero. Therefore, over each aperture, $\boldsymbol{M}^{TM_{10}^z} = -\boldsymbol{n} \times \boldsymbol{E}^{TM_{10}^z}$ in which $\boldsymbol{n}$ is the unit vector normal to that aperture. (14-a)

The $\boldsymbol{M}^{TM_{10}^z}$ over the two apertures at $y = 0$ and at $y = b$ do not contribute to the Far-Fields and, therefore, are not considered. (14-b)

Magnetic Current Density over the Aperture at $x = 0$

(14-a), (14-b) → $$\boldsymbol{M}^{TM_{10}^z}(x = 0) = j\omega A_0 \boldsymbol{a}_y \quad (14\text{-c})$$

Magnetic Current Density over the Aperture at $x = a$

$$\boldsymbol{M}^{TM_{10}^z}(x = a) = j\omega A_0 \boldsymbol{a}_y \quad (14\text{-d})$$

Eliminating the Ground Plane using the Image Theory

(14-c), (14-d) →
$$\boldsymbol{M}^{TM_{10}^z}(x = 0) = j2\omega A_0 \boldsymbol{a}_y \quad (15\text{-a})$$
$$\boldsymbol{M}^{TM_{10}^z}(x = a) = j2\omega A_0 \boldsymbol{a}_y \quad (15\text{-b})$$

(15) → It was demonstrated that the rectangular patch, in its dominant mode, can be modeled as a two-element array (same amplitude and phase) in which the basic element is a constant magnetic current over a rectangular aperture, i.e., $\boldsymbol{M}^{TM_{10}^z}(\boldsymbol{r}) = j2\omega A_0 \delta(x) \Pi\left(\frac{y}{b}\right) \Pi\left(\frac{z}{h}\right) \boldsymbol{a}_y$ and the array factor is given by $AF_2 = e^{jk\frac{a}{2}\boldsymbol{a}_r \cdot \boldsymbol{a}_x} + e^{-jk\frac{a}{2}\boldsymbol{a}_r \cdot \boldsymbol{a}_x}$. (16)

Far-Fields Associated with $\boldsymbol{M}^{TM_{10}^z}$

(16) → $$L_\theta = j2\omega([bh]A_0)\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\left[\frac{\sin\left(\frac{kh}{2}\cos\theta\right)}{\frac{kh}{2}\cos\theta}\right]\cos\theta\sin\phi \quad (17\text{-a})$$

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$$L_\phi = j2\omega([bh]A_0)\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\left[\frac{\sin\left(\frac{kh}{2}\cos\theta\right)}{\frac{kh}{2}\cos\theta}\right]\cos\phi \quad (17\text{-b})$$

(11-b), (17) → Imposing $\frac{h}{\lambda} \ll 1$ results in $\frac{\sin\left(\frac{kh}{2}\cos\theta\right)}{\frac{kh}{2}\cos\theta} \approx 1$ (18)

$$L_\theta = j2\omega([bh]A_0)\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\theta\sin\phi \quad (19\text{-a})$$

(17), (18) →

$$L_\phi = j2\omega([bh]A_0)\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\phi \quad (19\text{-b})$$

$$E_\theta = 2\omega k([bh]A_0)\frac{e^{-jkr}}{4\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\phi \quad (20\text{-a})$$

(1), (19) →

$$E_\phi = -2\omega k([bh]A_0)\frac{e^{-jkr}}{4\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\theta\sin\phi \quad (20\text{-b})$$

Array Factor

(16) → $$AF_2 = 2\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right) \quad (21)$$

(20), (21) →

$$E_\theta = 4\omega k([bh]A_0)\frac{e^{-jkr}}{4\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)\cos\phi$$

$$E_\phi = -4\omega k([bh]A_0)\frac{e^{-jkr}}{4\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}\sin\theta\sin\phi}\right]\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)\cos\theta\sin\phi$$

Circular Microstrip Patch

The circular patch is located at $z = \frac{h}{2}$ with a radius of ρ_0. The ground plane is located at $z = -\frac{h}{2}$. The space between the patch and the ground plane (height h) is filled with dielectric material. (22-a)
The perimeter of the patch (which is parallel to the $x-y$ plane) and the thickness of the dielectric material along the z axis define a surface which, here, is referred to as surface S.

The height h is assumed to be infinitesimally small, i.e., $\frac{h}{\lambda} \ll 1$. This implies that there are no variations in the fields along the z axis, i.e., $\frac{\partial}{\partial z} = 0$. Therefore, the fields underneath the patch (i.e., within the (22-b)

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thickness of the dielectric), which are denoted by $\boldsymbol{E}^{patch}$ and $\boldsymbol{H}^{patch}$, can be described using 2D modes.

The electric current density $\boldsymbol{J}$ over the surface of the patch represents the discontinuity in the magnetic fields on both sides of the patch. While $\boldsymbol{H}^{patch}$ is non-zero, the magnetic field above the patch is zero. Therefore, $\boldsymbol{J} = -\boldsymbol{a}_z \times \boldsymbol{H}^{patch}$. (22-c)

(22-b), (22-c) → The boundary condition imposed on the surface S must ensure that the electric current density over the patch does not have a normal component to the perimeter of the patch at the perimeter itself. In other words, $\boldsymbol{n}_b \cdot \boldsymbol{J} = 0$ in which $\boldsymbol{n}_b$ represents a unit vector normal to the perimeter at any point along the perimeter. (22-d)
From the above condition and (22-c) it follows that $H_\phi^{patch} = 0$ along the perimeter. This means that along the entire perimeter of the patch the tangential component of the magnetic field is zero. In other words, the surface S is a Perfect Magnetic Conductor (PMC).

(22-b), (22-c) → The modes that can exist underneath the patch can be determined by considering the components of the electric current density $\boldsymbol{J}$. From (22-c) it follows that $J_\rho = H_\phi^{patch}$ and that $J_\phi = -H_\rho^{patch}$. Therefore, for $\boldsymbol{J}$ to not wannish, $\boldsymbol{H}^{patch}$ must have at least one component in the direction of $\boldsymbol{a}_\rho$ or $\boldsymbol{a}_\phi$. The 2D TE^z mode which only has a magnetic field in the direction of $\boldsymbol{a}_z$ can't satisfy the above condition. Therefore, the only modes that can exist underneath the patch are 2D TM^z. (22-e)

The 2D TM^z Mode

$$\begin{aligned} E_\rho &= 0 & H_\rho &= \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} \\ E_\phi &= 0 & H_\phi &= -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} \\ E_z &= -j\omega A_z & H_z &= 0 \end{aligned} \quad (23\text{-a})$$

In the above equations, A_z satisfies the following PDE:
$\frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \omega^2 \mu\varepsilon A_z = 0.$ (23-b)

(22-d), (23-b) → $A_z^{mn} = A_0 J_m(k_\rho \rho)\cos(m\phi)$ in which $k_\rho = \frac{\chi'_{mn}}{\rho_0}$ and $k_\rho^2 = \omega^2\mu\varepsilon$ (23-c)

The Electric Field for the Dominant Mode TM_{11}^z

(23-a), (23-c) → $E_z^{TM_{11}^z} = -j\omega A_0 J_1\left(\chi'_{11}\frac{\rho}{\rho_0}\right)\cos\phi$ in which $f_r = \frac{\chi'_{11}}{2\pi\rho_0\sqrt{\mu\varepsilon}}$ (24)

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Magnetic Current Density

Surface S, as defined in (22-a), is a (circular) ribbon-shaped aperture with infinitesimal hight. The magnetic current density $\boldsymbol{M}^{TM_{11}^{z}}$ over this aperture represents the discontinuity in the electric fields on both sides of it. While $\boldsymbol{E}^{TM_{11}^{z}}$ is non-zero, the electric field beyond the patch is zero. Therefore, over the aperture, $\boldsymbol{M}^{TM_{11}^{z}} = -\boldsymbol{n} \times \boldsymbol{E}^{TM_{11}^{z}}$ in which $\boldsymbol{n}$ is the unit vector normal to the aperture. (25)

Magnetic Current Density over the Aperture

(24), (25) → $$\boldsymbol{M}^{TM_{11}^{z}}(\rho = \rho_0) = -j\omega A_0 J_1(\chi'_{11}) \cos\phi \, \boldsymbol{a}_\phi \quad (26)$$

Eliminating the Ground Plane using the Image Theory

(26) → $$\boldsymbol{M}^{TM_{11}^{z}}(\rho = \rho_0) = -j2\omega A_0 J_1(\chi'_{11}) \cos\phi \, \boldsymbol{a}_\phi \quad (27)$$

(27) → It was demonstrated that the circular patch, in its dominant mode, can be modeled as a circular ribbon with infinitesimal height carrying a circular sinusoidal magnetic current which is described by $\boldsymbol{M}^{TM_{11}^{z}}(\boldsymbol{r}) = -j2\omega A_0 J_1(\chi'_{11})\delta(\rho - \rho_0)\cos(\phi)\,\delta(z)\boldsymbol{a}_\phi$. (28)

Radiation Pattern of the Circular Sinusoidal Magnetic Current

Magnetic Current Density

$$\boldsymbol{M}(\boldsymbol{r}) = I_0^m \delta(\rho - \rho_0)\cos(\phi)\,\delta(z)\boldsymbol{a}_\phi$$

Far-Field Radiation Integral

$$L_\theta(\boldsymbol{r}) = -I_0^m(2\pi\rho_0)\left[\frac{J_1(k\rho_0 \sin\theta)}{k\rho_0 \sin\theta}\right]\cos\theta \sin\phi \quad (29\text{-a})$$

$$L_\phi(\boldsymbol{r}) = I_0^m(2\pi\rho_0)J'_1(k\rho_0 \sin\theta)\cos\phi \quad (29\text{-b})$$

Far-Fields

(1), (29) →

$$E_r = 0 \quad (30\text{-a})$$

$$E_\theta = -jk([2\pi\rho_0]I_0^m)\frac{e^{-jkr}}{4\pi r}J'_1(k\rho_0 \sin\theta)\cos\phi \quad (30\text{-b})$$

$$E_\phi = -jk([2\pi\rho_0]I_0^m)\frac{e^{-jkr}}{4\pi r}\left[\frac{J_1(k\rho_0 \sin\theta)}{k\rho_0 \sin\theta}\right]\cos\theta \sin\phi \quad (30\text{-c})$$

(28), (30) →

$$E_r = 0$$

$$E_\theta = -2\omega k J_1(\chi'_{11})([2\pi\rho_0]A_0)\frac{e^{-jkr}}{4\pi r}J'_1(k\rho_0 \sin\theta)\cos\phi$$

$$E_\phi = -2\omega k J_1(\chi'_{11})([2\pi\rho_0]A_0)\frac{e^{-jkr}}{4\pi r}\left[\frac{J_1(k\rho_0 \sin\theta)}{k\rho_0 \sin\theta}\right]\cos\theta \sin\phi$$

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8- Guiding Structures

Table 21. Derivation of the Dominant and the Higher-Order Modes of the Coaxial Cable

The Dominant Mode: TEM

$$\nabla^2\phi_e(\rho,\phi)=0 \quad (1\text{-a})$$

$$\boldsymbol{E}=\boldsymbol{E}_0 e^{-jkz} \text{ in which } \boldsymbol{E}_0=-\boldsymbol{\nabla}\phi_e \quad (1\text{-b})$$

$$\boldsymbol{H}=\frac{1}{\eta}\boldsymbol{a}_z\times\boldsymbol{E} \quad (1\text{-c})$$

Due to the symmetry of the coaxial structure, the electrostatic potential ϕ_e has no dependance on the cylindrical coordinate ϕ. (2)

(1-a), (2) → $$\phi_e(\rho)=C_1\ln(\rho)+D_1 \quad (3)$$

Boundary Conditions

$$\phi_e(\rho=\rho_1)=V_0 \quad (4\text{-a})$$

$$\phi_e(\rho=\rho_2)=0 \quad (4\text{-b})$$

(3), (4) → $$\phi_e(\rho)=\frac{V_0}{\ln\left(\frac{\rho_1}{\rho_2}\right)}\ln\left(\frac{\rho}{\rho_2}\right) \quad (5)$$

(1-b), (5) → $$\boldsymbol{E}_0=\frac{V_0}{\ln\left(\frac{\rho_2}{\rho_1}\right)}\frac{1}{\rho}\boldsymbol{a}_\rho$$

Higher-Order Modes

Homogeneous System of Linear Equations: $\boldsymbol{Ax}=\boldsymbol{0}$

Enforcing $\det(\boldsymbol{A})=0$, guarantees that the system has ***non-trivial*** solutions (i.e., $\boldsymbol{x}\neq\boldsymbol{0}$). Furthermore, the ***nullspace*** of $\boldsymbol{A}$ (i.e., any $\boldsymbol{x}\neq\boldsymbol{0}$ for which $\boldsymbol{Ax}=\boldsymbol{0}$) can be explored by the ***eigen decomposition*** of $\boldsymbol{A}$. In other words, any ***eigenvector*** of $\boldsymbol{A}$ for which the corresponding ***eigenvalue*** is zero, is in the nullspace of $\boldsymbol{A}$. (6)

TM^z

$$A_z^+(\rho,\phi,z)=A_0^+\left(C_1J_m(k_\rho\rho)+D_1Y_m(k_\rho\rho)\right)\cos(m\phi)\,e^{-jk_zz} \quad (7)$$

Boundary Conditions

$$E_\phi^+(\rho=\rho_1,\phi,z)=0 \quad (8\text{-a})$$

$$E_\phi^+(\rho=\rho_2,\phi,z)=0 \quad (8\text{-b})$$

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(7), (8) → $C_1 J_m(k_\rho \rho_1) + D_1 Y_m(k_\rho \rho_1) = 0$ (9-a)

$C_1 J_m(k_\rho \rho_2) + D_1 Y_m(k_\rho \rho_2) = 0$ (9-b)

(9) → $\begin{bmatrix} J_m(k_\rho \rho_1) & Y_m(k_\rho \rho_1) \\ J_m(k_\rho \rho_2) & Y_m(k_\rho \rho_2) \end{bmatrix} \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = 0$ (10)

(6), (10) → $J_m(k_\rho \rho_1) Y_m(k_\rho \rho_2) - Y_m(k_\rho \rho_1) J_m(k_\rho \rho_2) = 0$

TE^z

$$F_z^+(\rho, \phi, z) = F_0^+ \left(C_1 J_m(k_\rho \rho) + D_1 Y_m(k_\rho \rho) \right) \cos(m\phi)\, e^{-jk_z z} \quad (11)$$

Boundary Conditions

$E_\phi^+(\rho = \rho_1, \phi, z) = 0$ (12-a)

$E_\phi^+(\rho = \rho_2, \phi, z) = 0$ (12-b)

(11), (12) → $C_1 J'_m(k_\rho \rho_1) + D_1 Y'_m(k_\rho \rho_1) = 0$ (13-a)

$C_1 J'_m(k_\rho \rho_2) + D_1 Y'_m(k_\rho \rho_2) = 0$ (13-b)

(13) → $\begin{bmatrix} J'_m(k_\rho \rho_1) & Y'_m(k_\rho \rho_1) \\ J'_m(k_\rho \rho_2) & Y'_m(k_\rho \rho_2) \end{bmatrix} \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = 0$ (14)

(6), (14) → $J'_m(k_\rho \rho_1) Y'_m(k_\rho \rho_2) - Y'_m(k_\rho \rho_1) J'_m(k_\rho \rho_2) = 0$

Table 22. Derivation of the Excited TE Mode Amplitudes for a Rectangular Waveguide

Waveguide is oriented along the z axis with its cross-section on the $x - y$ plane. In a region of the waveguide, there exist sources $\boldsymbol{J}_1$, $\boldsymbol{M}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{M}_2$. The sources are confined to volume V bounded by surface S defined by the waveguide walls and two transverse planes S_{01} (with its normal vector pointing in the direction of the negative z axis) and S_{02} (with its normal vector pointing in the direction of the positive z axis). (1)

Lorentz Reciprocity Theorem

$$0 = \oiint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot d\boldsymbol{S} + \iiint_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dV \quad (2)$$

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$$0 = \iint_{S_{02}} \left[\left(E_{1x}H_{2y} - E_{1y}H_{2x}\right) - \left(E_{2x}H_{1y} - E_{2y}H_{1x}\right)\right] dS$$

(1), (2) $\rightarrow$

$$- \iint_{S_{01}} \left[\left(E_{1x}H_{2y} - E_{1y}H_{2x}\right) - \left(E_{2x}H_{1y} - E_{2y}H_{1x}\right)\right] dS \quad (3)$$

$$+ \iiint_{V} (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_2 \cdot \boldsymbol{M}_1 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2) dV$$

Assumption-I

$$\boldsymbol{J}_2 = \boldsymbol{0} \text{ and } \boldsymbol{M}_2 = \boldsymbol{0} \quad (4)$$

$$\iint_{S_{02}} \left[\left(E_{1x}H_{2y} - E_{1y}H_{2x}\right) - \left(E_{2x}H_{1y} - E_{2y}H_{1x}\right)\right] dS$$

(3), (4) $\rightarrow$

$$- \iint_{S_{01}} \left[\left(E_{1x}H_{2y} - E_{1y}H_{2x}\right) - \left(E_{2x}H_{1y} - E_{2y}H_{1x}\right)\right] dS \quad (5)$$

$$= \iiint_{V} (\boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_2 \cdot \boldsymbol{M}_1) dV$$

Forward Propagation of the TE Mode

Assumption-II

On S_{01}, $\boldsymbol{E}_1 = \boldsymbol{E}^-, \boldsymbol{H}_1 = \boldsymbol{H}^-, \boldsymbol{E}_2 = \boldsymbol{E}^-, \boldsymbol{H}_2 = \boldsymbol{H}^-$ (6-a)

On S_{02}, $\boldsymbol{E}_1 = \boldsymbol{E}^+, \boldsymbol{H}_1 = \boldsymbol{H}^+, \boldsymbol{E}_2 = \boldsymbol{E}^-, \boldsymbol{H}_2 = \boldsymbol{H}^-$ (6-b)

$$\iint_{S_{02}} \left[\left(E_x^+ H_y^- - E_y^+ H_x^-\right) - \left(E_x^- H_y^+ - E_y^- H_x^+\right)\right] dS$$

(5), (6) $\rightarrow$ (7)

$$= \iiint_{V} (\boldsymbol{E}^- \cdot \boldsymbol{J}_1 - \boldsymbol{H}^- \cdot \boldsymbol{M}_1) dV$$

For the TE mode of the rectangular waveguide, the relations between forward and backward propagating field amplitudes are as follows: $E_x^- = \frac{H_0^-}{H_0^+} E_x^+ e^{j2k_z z}$, $E_y^- = \frac{H_0^-}{H_0^+} E_y^+ e^{j2k_z z}$, $H_x^- = -\frac{H_0^-}{H_0^+} H_x^+ e^{j2k_z z}$, $H_y^- = -\frac{H_0^-}{H_0^+} H_y^+ e^{j2k_z z}$, $H_z^- = \frac{H_0^-}{H_0^+} H_z^+ e^{j2k_z z}$. (8)

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$$(7),(8) \quad \rightarrow \quad \iint_{S_{02}} \left[\left(-E_x^+ H_y^+ e^{j2k_z z} + E_y^+ H_x^+ e^{j2k_z z}\right) - \left(E_x^+ H_y^+ e^{j2k_z z} - E_y^+ H_x^+ e^{j2k_z z}\right)\right] dS \tag{9}$$

$$= \iiint_V \left(\left(\boldsymbol{E}^+ e^{j2k_z z}\right) \cdot \boldsymbol{J}_1 + \left(\boldsymbol{H}_{x,y}^+ e^{j2k_z z}\right) \cdot \boldsymbol{M}_{1x,y} - \left(H_z^+ e^{j2k_z z}\right) M_{1z} \right) dV$$

$$(9) \quad \rightarrow \quad 2 \iint_{S_{02}} \left[-E_x^+ H_y^+ e^{j2k_z z} + E_y^+ H_x^+ e^{j2k_z z}\right] dS \tag{10}$$

$$= \iiint_V \left(\left(\boldsymbol{E}^+ e^{j2k_z z}\right) \cdot \boldsymbol{J}_1 + \left(\boldsymbol{H}_{x,y}^+ e^{j2k_z z}\right) \cdot \boldsymbol{M}_{1x,y} - \left(H_z^+ e^{j2k_z z}\right) M_{1z} \right) dV$$

$$Z_{TE} \equiv \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} \tag{11}$$

$$- \frac{2}{Z_{TE}} \iint_{S_{02}} \left(E_x^+ E_x^+ e^{j2k_z z} + E_y^+ E_y^+ e^{j2k_z z}\right) dS \tag{12-a}$$

$$= \iiint_V \left(\left(\boldsymbol{E}^+ e^{j2k_z z}\right) \cdot \boldsymbol{J}_1 + \left(\boldsymbol{H}_{x,y}^+ e^{j2k_z z}\right) \cdot \boldsymbol{M}_{1x,y} - \left(H_z^+ e^{j2k_z z}\right) M_{1z} \right) dV$$

$$(10),(11) \quad \rightarrow$$

$$- 2Z_{TE} \iint_{S_{02}} \left(H_y^+ H_y^+ e^{j2k_z z} + H_x^+ H_x^+ e^{j2k_z z}\right) dS \tag{12-b}$$

$$= \iiint_V \left(\left(\boldsymbol{E}^+ e^{j2k_z z}\right) \cdot \boldsymbol{J}_1 + \left(\boldsymbol{H}_{x,y}^+ e^{j2k_z z}\right) \cdot \boldsymbol{M}_{1x,y} - \left(H_z^+ e^{j2k_z z}\right) M_{1z} \right) dV$$

Backward Propagation of the TE Mode

Assumption-III

$$\text{On } S_{01}, \boldsymbol{E}_1 = \boldsymbol{E}^-, \boldsymbol{H}_1 = \boldsymbol{H}^-, \boldsymbol{E}_2 = \boldsymbol{E}^+, \boldsymbol{H}_2 = \boldsymbol{H}^+ \tag{13-a}$$

$$\text{On } S_{02}, \boldsymbol{E}_1 = \boldsymbol{E}^+, \boldsymbol{H}_1 = \boldsymbol{H}^+, \boldsymbol{E}_2 = \boldsymbol{E}^+, \boldsymbol{H}_2 = \boldsymbol{H}^+ \tag{13-b}$$

$$(5),(13) \quad \rightarrow \quad - \iint_{S_{01}} \left[\left(E_x^- H_y^+ - E_y^- H_x^+\right) - \left(E_x^+ H_y^- - E_y^+ H_x^-\right)\right] dS \tag{14}$$

$$= \iiint_V \left(\boldsymbol{E}^+ \cdot \boldsymbol{J}_1 - \boldsymbol{H}^+ \cdot \boldsymbol{M}_1\right) dV$$

$$(8),(14) \quad \rightarrow \quad - \iint_{S_{01}} \left[\left(-E_x^- H_y^- e^{-j2k_z z} + E_y^- H_x^- e^{-j2k_z z}\right) - \left(E_x^- H_y^- e^{-j2k_z z} - E_y^- H_x^- e^{-j2k_z z}\right)\right] dS \tag{15}$$

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$$= \iiint_V \left((\boldsymbol{E}^- e^{-j2k_z z}) \cdot \boldsymbol{J}_1 + (\boldsymbol{H}^-_{x,y} e^{-j2k_z z}) \cdot \boldsymbol{M}_{1x,y} - (H^-_z e^{-j2k_z z}) M_{1z} \right) dV$$

$$- \quad 2 \iint_{S_{01}} \left[-E^-_x H^-_y e^{-j2k_z z} + E^-_y H^-_x e^{-j2k_z z} \right] dS$$

(15) → (16)

$$= \iiint_V \left((\boldsymbol{E}^- e^{-j2k_z z}) \cdot \boldsymbol{J}_1 + (\boldsymbol{H}^-_{x,y} e^{-j2k_z z}) \cdot \boldsymbol{M}_{1x,y} - (H^-_z e^{-j2k_z z}) M_{1z} \right) dV$$

$$- \quad \frac{2}{Z_{TE}} \iint_{S_{01}} \left[E^-_x E^-_x e^{-j2k_z z} + E^-_y E^-_y e^{-j2k_z z} \right] dS \qquad \text{(17-a)}$$

$$= \iiint_V \left((\boldsymbol{E}^- e^{-j2k_z z}) \cdot \boldsymbol{J}_1 + (\boldsymbol{H}^-_{x,y} e^{-j2k_z z}) \cdot \boldsymbol{M}_{1x,y} - (H^-_z e^{-j2k_z z}) M_{1z} \right) dV$$

(8), (11), (16) →

$$- \quad 2Z_{TE} \iint_{S_{01}} \left[H^-_y H^-_y e^{-j2k_z z} + H^-_x H^-_x e^{-j2k_z z} \right] dS \qquad \text{(17-b)}$$

$$= \iiint_V \left((\boldsymbol{E}^- e^{-j2k_z z}) \cdot \boldsymbol{J}_1 + (\boldsymbol{H}^-_{x,y} e^{-j2k_z z}) \cdot \boldsymbol{M}_{1x,y} - (H^-_z e^{-j2k_z z}) M_{1z} \right) dV$$

A Practical Formulation

Assumption-IV

From (12) and (17) it is evident that the expressions in the left-hand-side of these equations are independent of the z coordinate, because of the $e^{j2k_z z}$ factor in (12) and the $e^{-j2k_z z}$ factor in (17). Therefore, they can both be evaluated on a single constant-z plane which will henceforth be denoted as S_0. (18)

To simplify calculations as much as possible, S_0 is defined to be $z = 0$ plane and the volume V is situated such that S_{02} coincides with S_0.

(12), (17), (18) →

$$\frac{2}{Z_{TE}} \iint_{S_0} \left(E^+_x E^+_x + E^+_y E^+_y \right) dS = - \iiint_V \left((\boldsymbol{E}^+ e^{+j2k_z z}) \cdot \boldsymbol{J} + (\boldsymbol{H}^+_{x,y} e^{+j2k_z z}) \cdot \boldsymbol{M}_{x,y} - (H^+_z e^{+j2k_z z}) M_z \right) dV$$

$$2Z_{TE} \iint_{S_0} \left(H^+_x H^+_x + H^+_y H^+_y \right) dS = - \iiint_V \left((\boldsymbol{E}^+ e^{+j2k_z z}) \cdot \boldsymbol{J} + (\boldsymbol{H}^+_{x,y} e^{+j2k_z z}) \cdot \boldsymbol{M}_{x,y} - (H^+_z e^{+j2k_z z}) M_z \right) dV$$

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Table 23. Derivation of the Excited Dominant Mode Amplitudes for Rectangular Waveguide

TE_{10} Mode Fields

Forward Propagation

$$E_y^+ = A^+ \sin\left(\frac{\pi}{a}x\right) e^{-jk_z z} \tag{1-a}$$

$$H_x^+ = -\frac{A^+}{Z_{TE_{10}}} \sin\left(\frac{\pi}{a}x\right) e^{-jk_z z} \tag{1-b}$$

$$H_z^+ = j\frac{\pi A^+}{ak_z Z_{TE_{10}}} \cos\left(\frac{\pi}{a}x\right) e^{-jk_z z} \tag{1-c}$$

Backward Propagation

$$E_y^- = A^- \sin\left(\frac{\pi}{a}x\right) e^{jk_z z} \tag{2-a}$$

$$H_x^- = \frac{A^-}{Z_{TE_{10}}} \sin\left(\frac{\pi}{a}x\right) e^{jk_z z} \tag{2-b}$$

$$H_z^- = j\frac{\pi A^-}{ak_z Z_{TE_{10}}} \cos\left(\frac{\pi}{a}x\right) e^{jk_z z} \tag{2-c}$$

The Practical Formula

$$\frac{2}{Z_{TE}} \iint_{S_0} \left(E_x^{\pm} E_x^{\pm} + E_y^{\pm} E_y^{\pm}\right) dS = -\iiint_V \left(\left(\boldsymbol{E}^{\pm} e^{\pm j2k_z z}\right) \cdot \boldsymbol{J} + \left(\boldsymbol{H}_{x,y}^{\pm} e^{\pm j2k_z z}\right) \cdot \boldsymbol{M}_{x,y} - \left(H_z^{\pm} e^{\pm j2k_z z}\right) M_z\right) dV \tag{3}$$

In the above formula, S_0 is the $z = 0$ plane.

Excitation with Center-Fed, Electric Current Probe

$$\boldsymbol{J} = I_0 \delta\left(x - \frac{a}{2}\right) \frac{\sin k(h-y)}{\sin kh} \delta(z) \boldsymbol{a}_y \tag{4}$$

$$\frac{2}{Z_{TE_{10}}} \iint_{S_0} E_y^+ E_y^+ dS = \frac{ab}{Z_{TE_{10}}} (A^+)^2 \tag{5-a}$$

(1-a), (3), (4) $\rightarrow$

$$-\iiint_V \left(\boldsymbol{E}^+ e^{j2k_z z}\right) \cdot \boldsymbol{J} dV = -A^+ \left(\frac{1 - \cos kh}{k \sin kh}\right) I_0 \tag{5-b}$$

(5) $\rightarrow$ $A^+ = -\frac{Z_{TE_{10}}}{ab} \left(\frac{1 - \cos kh}{k \sin kh}\right) I_0$

Excitation with an Aperture in the Broad Wall

Electric and Magnetic Polarizabilities of an Aperture

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$$\boldsymbol{P}_e \equiv \left(\varepsilon_0 \alpha_e \delta(\boldsymbol{r}-\boldsymbol{r}_0)(\boldsymbol{n}\cdot\boldsymbol{E})\right)\boldsymbol{n} \tag{6-a}$$

For a round hole, $\alpha_e = \frac{2r^3}{3}$.

$$\boldsymbol{P}_m \equiv \left(-\alpha_m \delta(\boldsymbol{r}-\boldsymbol{r}_0)\right)(\boldsymbol{H}-(\boldsymbol{n}\cdot\boldsymbol{H})\boldsymbol{n}) \tag{6-b}$$

For a round hole, $\alpha_m = \frac{4r^3}{3}$.

Current Density Equivalents of Aperture Polarizabilities

$$\boldsymbol{J} = j\omega \boldsymbol{P}_e \tag{7-a}$$

$$\boldsymbol{M} = j\omega\mu_0 \boldsymbol{P}_m \tag{7-b}$$

A round hole in the broad wall of the waveguide is located at $(x = s, y = 0, z = 0)$. It is excited by the forward propagation of the dominant mode, with the amplitude A, in a rectangular waveguide positioned beneath it.
This hole can be modeled by replacing it with both electric and magnetic current densities. (8)
It is worth reiterating that the E_y, H_x and H_z used in the construction of these current densities refer to the forward propagating dominant mode in the waveguide beneath.

(6), (7), (8) →

$$\boldsymbol{J} = \left(j\omega\varepsilon_0\alpha_e\delta(x-s)\delta(y)\delta(z)E_y\right)\boldsymbol{a}_y \tag{9-a}$$

$$\boldsymbol{M} = \left(-j\omega\mu_0\alpha_m\delta(x-s)\delta(y)\delta(z)H_x\right)\boldsymbol{a}_x + \left(-j\omega\mu_0\alpha_m\delta(x-s)\delta(y)\delta(z)H_z\right)\boldsymbol{a}_z \tag{9-b}$$

$$\frac{2}{Z_{TE_{10}}}\iint_{S_0} E_y^+ E_y^+ dS = \frac{ab}{Z_{TE_{10}}}(A^+)^2 \tag{10-a}$$

$$-\iiint_V \left(\boldsymbol{E}^+ e^{j2k_z z}\right)\cdot\boldsymbol{J}dV = -j\omega\varepsilon_0\alpha_e\left(\sin\frac{\pi}{a}s\right)^2 AA^+ \tag{10-b}$$

(1), (3), (9) →

$$-\iiint_V \left(\boldsymbol{H}_{x,y}^+ e^{j2k_z z}\right)\cdot\boldsymbol{M}_{x,y}dV = j\omega\frac{\mu_0\alpha_m}{Z_{TE_{10}}^2}\left(\sin\frac{\pi}{a}s\right)^2 AA^+ \tag{10-c}$$

$$\iiint_V \left(H_z^+ e^{j2k_z z}\right)M_z dV = j\omega\frac{\mu_0\alpha_m\pi^2}{a^2k_z^2Z_{TE_{10}}^2}\left(\cos\frac{\pi}{a}s\right)^2 AA^+ \tag{10-d}$$

(3), (10) →

$$A^+ = -j\frac{\omega Z_{TE_{10}}}{ab}\left(\varepsilon_0\alpha_e\left(\sin\frac{\pi}{a}s\right)^2 - \frac{\mu_0\alpha_m}{Z_{TE_{10}}^2}\left(\left(\sin\frac{\pi}{a}s\right)^2 + \frac{\pi^2}{a^2k_z^2}\left(\cos\frac{\pi}{a}s\right)^2\right)\right)A$$

$$\frac{2}{Z_{TE_{10}}}\iint_{S_0} E_y^- E_y^- dS = \frac{ab}{Z_{TE_{10}}}(A^-)^2 \tag{11-a}$$

(2), (3), (9) →

$$-\iiint_V \left(\boldsymbol{E}^- e^{-j2k_z z}\right)\cdot\boldsymbol{J}dV = -j\omega\varepsilon_0\alpha_e\left(\sin\frac{\pi}{a}s\right)^2 AA^- \tag{11-b}$$

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$$-\iiint_V \left(\boldsymbol{H}^-_{x,y} e^{-j2k_z z}\right) \cdot \boldsymbol{M}_{x,y} dV = -j\omega \frac{\mu_0 \alpha_m}{Z^2_{TE_{10}}} \left(\sin\frac{\pi}{a}s\right)^2 AA^- \tag{11-c}$$

$$\iiint_V \left(H^-_z e^{-j2k_z z}\right) M_z dV = j\omega \frac{\mu_0 \alpha_m \pi^2}{a^2 k_z^2 Z^2_{TE_{10}}} \left(\cos\frac{\pi}{a}s\right)^2 AA^- \tag{11-d}$$

$$(11) \quad \rightarrow \quad A^- = -j\frac{\omega Z_{TE_{10}}}{ab}\left(\varepsilon_0 \alpha_e \left(\sin\frac{\pi}{a}s\right)^2 + \frac{\mu_0 \alpha_m}{Z^2_{TE_{10}}}\left(\left(\sin\frac{\pi}{a}s\right)^2 - \frac{\pi^2}{a^2 k_z^2}\left(\cos\frac{\pi}{a}s\right)^2\right)\right) A$$

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Table 24. Derivation of the Bandwidth for the Quarter-Wave and Multi-Section Transformers

$$BW \equiv \frac{\Delta f}{f_0} \tag{1}$$

$$\theta \equiv \beta l = \left(\frac{2\pi}{\lambda_g}\right) l \tag{2}$$

(2) → For any section, the length l_n is taken to be a quarter of a wavelength at the center (operating) frequency, i.e., $l_n = \frac{\lambda_{gn}^{@f_0}}{4}$ which is equivalent to an electrical length of $\theta_n^{@f_0} = \frac{\pi}{2}$. As such, its electrical length at any frequency is given by $\theta_n^{@f} = \frac{\pi}{2}\frac{\lambda_{gn}^{@f_0}}{\lambda_{gn}^{@f}}$ or equivalently, $\theta_n^{@f} = \frac{\pi}{2}\frac{f}{f_0}$. (3)

(3) → $$\frac{\Delta f}{f_0} = \frac{2}{\pi}\Delta\theta \tag{4}$$

With the assumption that the bandwidth is symmetric with respect to the center point $\theta = \frac{\pi}{2}$, it follows that $\theta_1 = \theta_m$ and $\theta_2 = \pi - \theta_m$. Therefore, $\Delta\theta = \pi - 2\theta_m$. (5)

(1), (4), (5) → $$BW = 2 - \frac{4}{\pi}\theta_m \tag{6}$$

Quarter-Wave Transformer

$$Z_{IN} = Z_1 \frac{Z_L + jZ_1 \tan\theta}{Z_1 + jZ_L \tan\theta} \tag{7}$$

$$Z_1 = \sqrt{Z_0 Z_L} \tag{8-a}$$

$$|\Gamma_{IN}(\theta)| \leq \Gamma_m \tag{8-b}$$

(7) →
$$\Gamma_{IN}(\theta) = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} = \frac{Z_1(Z_L - Z_0) + j(Z_1^2 - Z_0 Z_L)\tan\theta}{Z_1(Z_L + Z_0) + j(Z_1^2 + Z_0 Z_L)\tan\theta} \tag{9}$$

(8-a), (9) → $$\Gamma_{IN}(\theta) = \frac{Z_L - Z_0}{Z_L + Z_0 + j2\sqrt{Z_0 Z_L}\tan\theta} \tag{10}$$

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$$|\Gamma_{IN}(\theta)| = \frac{|Z_L - Z_0|}{\left|Z_L + Z_0 + j2\sqrt{Z_0 Z_L}\tan\theta\right|}$$

$$= \frac{|Z_L - Z_0|}{\sqrt{(Z_L + Z_0)^2 + 4Z_0 Z_L(\tan\theta)^2}}$$

(10) → $$= \frac{|Z_L - Z_0|}{\sqrt{(Z_L - Z_0)^2 + \frac{4Z_0 Z_L}{(\cos\theta)^2}}} \quad (11)$$

$$= \frac{1}{\sqrt{\left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0}\right)^2 + (\cos\theta)^2}}|\cos\theta|$$

(8-b), (11) → $$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}$$

Multi-Section Transformers

Boundary Conditions

$$\Gamma(\theta = 0) = \Gamma_L \quad (12\text{-a})$$

$$|\Gamma(\theta = \theta_m)| = \Gamma_m; \text{ This ensures that } |\Gamma(\theta)| \le \Gamma_m. \quad (12\text{-b})$$

Binomial Design

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N \quad (13)$$

(12-a), (13) → $$A = \frac{1}{2^N}\Gamma_L \quad (14)$$

(12-b), (13), (14) → $$\cos\theta_m = \left(\frac{\Gamma_m}{|\Gamma_L|}\right)^{\frac{1}{N}}$$

Chebyshev Design

$$\Gamma(\theta) = e^{-jN\theta}[AT_N(\sec\theta_m \cos\theta)] \quad (15)$$

$$T_n(x) = \cos(n\cos^{-1}x) \quad \text{for } |x| < 1 \quad (16\text{-a})$$

$$T_n(x) = \cosh(n\cosh^{-1}x) \quad \text{for } x > 1 \quad (16\text{-b})$$

(12-a), (15) → $$A = \frac{1}{T_N(\sec\theta_m)}\Gamma_L \quad (17)$$

(12-b), (15), (17) → $$T_N(\sec\theta_m) = \frac{|\Gamma_L|}{\Gamma_m} \quad (18)$$

(16-b), (18) → $$\sec\theta_m = \cosh\left(\frac{1}{N}\cosh^{-1}\left(\frac{|\Gamma_L|}{\Gamma_m}\right)\right)$$

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(17), (18) → $A = \left(\frac{\Gamma_L}{|\Gamma_L|}\right)\Gamma_m$

Table 25. Derivation of the Wilkinson Divider Design Equations

Properties of the Wilkinson Divider

$$\frac{P_3}{P_2} = K^2 \quad \text{(1-a)}$$

Voltage at ports two and three are equal. (1-b)

Ports two and three are isolated. (1-c)

$$V_3^- = V_2^- \quad \text{(2)}$$

Available Power Split at the Junction

$$P_2 = \frac{1}{2}\left|V_{n,1}^+\right|^2 = \frac{1}{2}\frac{1}{Z_{IN,2}}|V_1^+|^2 \quad \text{(3-a)}$$

$$P_3 = \frac{1}{2}\left|V_{n,1}^+\right|^2 = \frac{1}{2}\frac{1}{Z_{IN,3}}|V_1^+|^2 \quad \text{(3-b)}$$

(1-a), (3) → $Z_{IN,3} = \frac{1}{K^2}Z_{IN,2}$ (4)

Available Power at Ports Two and Three

$$P_2 = \frac{1}{2}\left|V_{n,2}^-\right|^2 = \frac{1}{2}\frac{1}{Z_{L,2}}|V_2^-|^2 \quad \text{(5-a)}$$

$$P_3 = \frac{1}{2}\left|V_{n,3}^-\right|^2 = \frac{1}{2}\frac{1}{Z_{L,3}}|V_3^-|^2 \quad \text{(5-b)}$$

(1-a), (2), (5) → $Z_{L,3} = \frac{1}{K^2}Z_{L,2}$ (6)

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Quarter-Wave Transformer

$$\frac{Z_{IN}}{Z_0} = \left(\frac{Z_L}{Z_0}\right)^{-1} \tag{7}$$

(7) → $$Z_{IN,2} = \frac{Z_2^2}{Z_{L,2}} \tag{8-a}$$

(7) → $$Z_{IN,3} = \frac{Z_3^2}{Z_{L,3}} \tag{8-b}$$

(4), (6), (8) → $$Z_3 = \frac{1}{K^2} Z_2 \tag{9}$$

Matched Port One

$$\frac{1}{Z_0} = \frac{1}{Z_{IN,2}} + \frac{1}{Z_{IN,3}} \tag{10}$$

(8), (10) → $$\frac{1}{Z_0} = \frac{Z_{L,2}}{Z_2^2} + \frac{Z_{L,3}}{Z_3^2} \tag{11}$$

(6), (9), (11) → $$Z_{L,2} = \left(\frac{1}{1+K^2}\right)\frac{Z_2^2}{Z_0}$$

Table 26. Derivation of the Scattering Matrix of a Coupled-Line Coupler

Even & Odd Mode Analysis of the Four-Port with Two Symmetry Axes

The Scattering Matrix

$$\begin{bmatrix} V_{n,1}^- \\ V_{n,2}^- \\ V_{n,3}^- \\ V_{n,4}^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} V_{n,1}^+ \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

(1) → $$V_{n,1}^- = S_{11} V_{n,1}^+ \tag{2-a}$$

(1) → $$V_{n,2}^- = S_{21} V_{n,1}^+ \tag{2-b}$$

(1) → $$V_{n,3}^- = S_{31} V_{n,1}^+ \tag{2-c}$$

(1) → $$V_{n,4}^- = S_{41} V_{n,1}^+ \tag{2-d}$$

The Even & Odd Mode Decomposition

$$\boldsymbol{V}_n^+ = \begin{bmatrix} V_{n,1}^+ \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

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$$= \begin{bmatrix} V_0^+ \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{V_0^+}{4}\left(\begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \end{bmatrix} + \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix} + \begin{bmatrix} +1 \\ +1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} +1 \\ -1 \\ -1 \\ +1 \end{bmatrix}\right)$$

Even-Even Mode Outputs

$$\boldsymbol{V}_a^+ = \frac{V_0^+}{4}\begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \end{bmatrix}$$

$$V_{a,1}^- = \Gamma_{a,1} V_{a,1}^+ = \left(+\frac{1}{4}\Gamma_a\right) V_0^+ \quad (4\text{-a})$$

(3) → $$V_{a,2}^- = \Gamma_{a,2} V_{a,2}^+ = \left(+\frac{1}{4}\Gamma_a\right) V_0^+ \quad (4\text{-b})$$

$$V_{a,3}^- = \Gamma_{a,3} V_{a,3}^+ = \left(+\frac{1}{4}\Gamma_a\right) V_0^+ \quad (4\text{-c})$$

$$V_{a,4}^- = \Gamma_{a,4} V_{a,4}^+ = \left(+\frac{1}{4}\Gamma_a\right) V_0^+ \quad (4\text{-d})$$

$$Z_{IN} = -jZ_{0e}\cot\left(\beta_e \frac{l}{2}\right) \quad (5\text{-a})$$

$$\Gamma_a \equiv \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0}$$

(5-a) → $$= \frac{-jZ_{0e}\cot\left(\beta_e \frac{l}{2}\right) - Z_0}{-jZ_{0e}\cot\left(\beta_e \frac{l}{2}\right) + Z_0} \quad (5\text{-b})$$

Even-Odd Mode Outputs

(3) → $$\boldsymbol{V}_b^+ = \frac{V_0^+}{4}\begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix}$$

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$$V_{b,1}^- = \Gamma_{b,1} V_{b,1}^+ = \left(+\frac{1}{4}\Gamma_b\right) V_0^+ \quad (6\text{-a})$$

$$V_{b,2}^- = \Gamma_{b,2} V_{b,2}^+ = \left(-\frac{1}{4}\Gamma_b\right) V_0^+ \quad (6\text{-b})$$

$$V_{b,3}^- = \Gamma_{b,3} V_{b,3}^+ = \left(+\frac{1}{4}\Gamma_b\right) V_0^+ \quad (6\text{-c})$$

$$V_{b,4}^- = \Gamma_{b,4} V_{b,4}^+ = \left(-\frac{1}{4}\Gamma_b\right) V_0^+ \quad (6\text{-d})$$

$$Z_{IN} = jZ_{0e} \tan\left(\beta_e \frac{l}{2}\right) \quad (7\text{-a})$$

$$\Gamma_b \equiv \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} \quad (7\text{-b})$$

(7-a) →

$$= \frac{jZ_{0e} \tan\left(\beta_e \frac{l}{2}\right) - Z_0}{jZ_{0e} \tan\left(\beta_e \frac{l}{2}\right) + Z_0}$$

Odd-Even Mode Outputs

$$\boldsymbol{V}_c^+ = \frac{V_0^+}{4}\begin{bmatrix} +1 \\ +1 \\ -1 \\ -1 \end{bmatrix}$$

$$V_{c,1}^- = \Gamma_{c,1} V_{c,1}^+ = \left(+\frac{1}{4}\Gamma_c\right) V_0^+ \quad (8\text{-a})$$

(3) →

$$V_{c,2}^- = \Gamma_{c,2} V_{c,2}^+ = \left(+\frac{1}{4}\Gamma_c\right) V_0^+ \quad (8\text{-b})$$

$$V_{c,3}^- = \Gamma_{c,3} V_{c,3}^+ = \left(-\frac{1}{4}\Gamma_c\right) V_0^+ \quad (8\text{-c})$$

$$V_{c,4}^- = \Gamma_{c,4} V_{c,4}^+ = \left(-\frac{1}{4}\Gamma_c\right) V_0^+ \quad (8\text{-d})$$

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$$Z_{IN} = -jZ_{0o}\cot\left(\beta_o\frac{l}{2}\right) \tag{9-a}$$

$$\Gamma_c \equiv \frac{Z_{IN}-Z_0}{Z_{IN}+Z_0}$$

(9-a) →

$$= \frac{-jZ_{0o}\cot\left(\beta_o\frac{l}{2}\right)-Z_0}{-jZ_{0o}\cot\left(\beta_o\frac{l}{2}\right)+Z_0} \tag{9-b}$$

Odd-Odd Mode Outputs

$$\boldsymbol{V}_d^+ = \frac{V_0^+}{4}\begin{bmatrix}+1\\-1\\-1\\+1\end{bmatrix}$$

$$V_{d,1}^- = \Gamma_{d,1}V_{d,1}^+ = \left(+\frac{1}{4}\Gamma_d\right)V_0^+ \tag{10-a}$$

(3) →

$$V_{d,2}^- = \Gamma_{d,2}V_{d,2}^+ = \left(-\frac{1}{4}\Gamma_d\right)V_0^+ \tag{10-b}$$

$$V_{d,3}^- = \Gamma_{d,3}V_{d,3}^+ = \left(-\frac{1}{4}\Gamma_d\right)V_0^+ \tag{10-c}$$

$$V_{d,4}^- = \Gamma_{d,4}V_{d,4}^+ = \left(+\frac{1}{4}\Gamma_d\right)V_0^+ \tag{10-d}$$

$$Z_{IN} = jZ_{0o}\tan\left(\beta_o\frac{l}{2}\right) \tag{11-a}$$

$$\Gamma_d \equiv \frac{Z_{IN}-Z_0}{Z_{IN}+Z_0}$$

(11-a) →

$$= \frac{jZ_{0o}\tan\left(\beta_o\frac{l}{2}\right)-Z_0}{jZ_{0o}\tan\left(\beta_o\frac{l}{2}\right)+Z_0} \tag{11-b}$$

(3), (4-a), (6-a), (8-a), (10-a) →

$$V_{n,1}^- = V_{a,1}^- + V_{b,1}^- + V_{c,1}^- + V_{d,1}^- = \frac{1}{4}(\Gamma_a+\Gamma_b+\Gamma_c+\Gamma_d)V_0^+ \tag{12-a}$$

(3), (4-b), (6-b), (8-b), (10-b) →

$$V_{n,2}^- = V_{a,2}^- + V_{b,2}^- + V_{c,2}^- + V_{d,2}^- = \frac{1}{4}(\Gamma_a-\Gamma_b+\Gamma_c-\Gamma_d)V_0^+ \tag{12-b}$$

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(3), (4-c), (6-c), (8-c), (10-c) → $V_{n,3}^- = V_{a,3}^- + V_{b,3}^- + V_{c,3}^- + V_{d,3}^- = \frac{1}{4}(\Gamma_a + \Gamma_b - \Gamma_c - \Gamma_d)V_0^+$ (12-c)

(3), (4-d), (6-d), (8-d), (10-d) → $V_{n,4}^- = V_{a,4}^- + V_{b,4}^- + V_{c,4}^- + V_{d,4}^- = \frac{1}{4}(\Gamma_a - \Gamma_b - \Gamma_c + \Gamma_d)V_0^+$ (12-d)

(2-a), (12-a) → $S_{11} = \frac{1}{4}(\Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d)$ (13-a)

(2-b), (12-b) → $S_{21} = \frac{1}{4}(\Gamma_a - \Gamma_b + \Gamma_c - \Gamma_d)$ (13-b)

(2-c), (12-c) → $S_{31} = \frac{1}{4}(\Gamma_a + \Gamma_b - \Gamma_c - \Gamma_d)$ (13-c)

(2-d), (12-d) → $S_{41} = \frac{1}{4}(\Gamma_a - \Gamma_b - \Gamma_c + \Gamma_d)$ (13-d)

The Scattering Parameters of the Four-Port

Important Conditions

$\beta_e = \beta$ (14-a)

$\beta_o = \beta$ (14-b)

$Z_{0e}Z_{0o} = Z_0^2$ (14-c)

(14-a), (14-b) → $\theta' \equiv \frac{\beta l}{2}$ (15-a)

$C \equiv \frac{Z_{0e} - Z_{0o}}{Z_{0e} + Z_{0o}}$ (15-b)

(5-b), (7-b), (14), (15) → $\Gamma_a + \Gamma_b = 2\frac{C}{1 - j\sqrt{1 - C^2}\cot 2\theta'}$ (16-a)

(5-b), (9-b), (14), (15) → $\Gamma_a + \Gamma_c = 2\frac{\sqrt{1 - C^2}}{\sqrt{1 - C^2}\cos 2\theta' + j\sin 2\theta'}$ (16-b)

(5-b), (11-b), (14), (15) → $\Gamma_a + \Gamma_d = 0$ (16-c)

(7-b), (9-b), (14), (15) → $\Gamma_b + \Gamma_c = 0$ (16-d)

(7-b), (11-b), (14), (15) → $\Gamma_b + \Gamma_d = -2\frac{\sqrt{1 - C^2}}{\sqrt{1 - C^2}\cos 2\theta' + j\sin 2\theta'}$ (16-e)

(9-b), (11-b), (14), (15) → $\Gamma_c + \Gamma_d = -2\frac{C}{1 - j\sqrt{1 - C^2}\cot 2\theta'}$ (16-f)

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(13-a), (16-a), (16-f) → $S_{11} = 0$ (17-a)

(13-b), (16-b), (16-e) → $S_{21} = \dfrac{\sqrt{1-C^2}}{\sqrt{1-C^2}\cos 2\theta' + j\sin 2\theta'}$ (17-b)

(13-c), (16-a), (16-f) → $S_{31} = \dfrac{C}{1 - j\sqrt{1-C^2}\cot 2\theta'}$ (17-c)

(13-d), (16-c), (16-d) → $S_{41} = 0$ (17-d)

Maximum of $|S_{31}|$ is obtained when $2\theta' = \frac{\pi}{2}$; Or, equivalently, when $l = \frac{\lambda_g}{4}$. (18)

(17), (18) →
$S_{11} = 0$
$S_{21} = -j\sqrt{1-C^2}$
$S_{31} = C$
$S_{41} = 0$

Low-Coupling Approximation of S_{31}

$\theta \equiv 2\theta'$ (19)

(17-c), (19) → $S_{31}(\theta) \approx jC\sin\theta\, e^{-j\theta}$

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Part IV

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10- Finite Difference Method (FDM)

Table 27. Derivation of the Discretized Form of the Laplacian Operator

$$f(x+\Delta) = f(x) + \Delta f'(x) + \frac{1}{2}\Delta^2 f''(x) + \cdots \quad \text{(TE)}$$

(TE) $\rightarrow$

$$f(x+h,y) \approx f(x,y) + h\frac{\partial f(x,y)}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 f(x)}{\partial x^2} \quad \text{(1-a)}$$

$$f(x-h,y) \approx f(x,y) - h\frac{\partial f(x,y)}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 f(x)}{\partial x^2} \quad \text{(1-b)}$$

(1) $\rightarrow$

$$\frac{\partial^2 f(x)}{\partial x^2} \approx \frac{1}{h^2}[f(x+h,y) + f(x-h,y) - 2f(x,y)] \quad (2)$$

(TE) $\rightarrow$

$$f(x,y+h) \approx f(x,y) + h\frac{\partial f(x,y)}{\partial y} + \frac{1}{2}h^2\frac{\partial^2 f(x)}{\partial y^2} \quad \text{(3-a)}$$

$$f(x,y-h) \approx f(x,y) - h\frac{\partial f(x,y)}{\partial y} + \frac{1}{2}h^2\frac{\partial^2 f(x)}{\partial y^2} \quad \text{(3-b)}$$

(3) $\rightarrow$

$$\frac{\partial^2 f(x)}{\partial y^2} \approx \frac{1}{h^2}[f(x,y+h) + f(x,y-h) - 2f(x,y)] \quad (4)$$

(2), (4) $\rightarrow$

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} + \frac{\partial^2 f(x)}{\partial y^2} &\approx \frac{1}{h^2}f(x+h,y) + \frac{1}{h^2}f(x-h,y) \\ &+ \frac{1}{h^2}f(x,y+h) + \frac{1}{h^2}f(x,y-h) \\ &- \frac{4}{h^2}f(x,y)\end{aligned}$$

Table 28. Derivation of the Analytic Solution for the Potential Box

$$\begin{aligned}\nabla^2 V(x,y) &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \\ &= 0\end{aligned} \quad (1)$$

$$V(x,y) = f(x)g(y) \quad (2)$$

$$f(x) = C_1\cos k_x x + D_1 \sin k_x x \quad \text{(3-a)}$$

$$g(y) = C_2\cos k_y y + D_2 \sin k_y y \quad \text{(3-b)}$$

(1), (2), (3) $\rightarrow$

$$V(x,y) = (C_1\cos k_x x + D_1 \sin k_x x)\left(C_2\cos k_y y + D_2 \sin k_y y\right) \quad \text{(4-a)}$$

$$k_x^2 + k_y^2 = 0 \quad \text{(4-b)}$$

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$$k_x = jk \tag{5-a}$$
$$k_y = k \tag{5-b}$$

(4-a), (5) → $$V(x,y) = (C_1 \cosh kx + D_1' \sinh kx)(C_2 \cos ky + D_2 \sin ky) \tag{6}$$

Boundary Conditions

$$V(0,y) = V_0 \tag{7-a}$$
$$V(a,y) = 0 \tag{7-b}$$
$$V(x,0) = 0 \tag{7-c}$$
$$V(x,b) = 0 \tag{7-d}$$

(6), (7-b) → $$C_1 \cosh ka + D_1' \sinh ka = 0 \tag{8}$$

(8) → $$C_1 = -\frac{D_1' \sinh ka}{\cosh ka} \tag{9}$$

(6), (9) → $$V(x,y) = C' \sinh k(a-x)\,(C_2 \cos ky + D_2 \sin ky) \tag{10}$$

(7-c), (7-d), (10) → $$V_n(x,y) = C_n'' \sinh k_n(a-x) \sin k_n y \qquad k_n = n\frac{\pi}{b} \tag{11}$$

(7-a), (11) →
$$V_0 = \sum_{n=1}^{\infty} V_n(0,y) = \sum_{n=1}^{\infty} C_n'' \sinh\left(n\frac{\pi}{b}a\right) \sin\left(n\frac{\pi}{b}y\right) \tag{12}$$

(12) → $$C_n'' = \frac{4V_0}{n\pi \sinh n\frac{\pi}{b}a} \qquad n \in Odd \tag{13}$$

(11), (12), (13) → $$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{n\left(\sinh n\frac{\pi}{b}a\right)} \sinh n\frac{\pi}{b}(a-x) \sin n\frac{\pi}{b}y \qquad n \in Odd$$

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11- Finite Element Method (FEM)

Table 29. Derivation of the 2D Linear Element Model

Weighted-Residual Method

To find the solution of $\mathcal{L}(\psi) = f$ over the domain Ω, in which $\mathcal{L}(\cdot)$ is a ***linear differential operator***, the domain of interest is divided into non-overlapping segments Ω_e^i over which the unknow solution ψ is approximated by the sum of a set of known ***approximation functions*** ϕ_j each scaled by an unknown coefficient ψ_j. This scheme transforms the original problem to that of constructing a system of linear equations for the scaling factors in the entire domain Ω.

A system of linear equations can be set up by setting $\int_{\Omega_e^i} w_k R \, d\Omega = 0$ over each Ω_e^i in which $R \equiv \mathcal{L}(\psi) - f$ is called the ***residual*** and w_k are called the ***weighting*** (testing) functions. In the ***Galerkin*** variant of this method, w_k are chosen to be the same as ϕ_j.

Continuity of Partial Derivatives

The second order derivatives are expressed as $\frac{\partial}{\partial u}\left(\alpha \frac{\partial \psi}{\partial u}\right)$ instead of $\frac{\partial^2 \psi}{\partial u^2}$ to emphasize that at the boundary between two different media, $\alpha \frac{\partial \psi}{\partial u}$ are continuous not $\frac{\partial \psi}{\partial u}$. More specifically, at the boundary between two segments Ω_e^1 and Ω_e^2 with respective parameters α_1 and α_2, the following condition is satisfied $(\alpha_1 \boldsymbol{\nabla}\psi_1) \cdot \boldsymbol{n} = (\alpha_2 \boldsymbol{\nabla}\psi_2) \cdot \boldsymbol{n}$.
This means that, in 2D time-harmonic problems, $\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}$ for the TE^z mode and $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0}$ for the TM^z mode are automatically satisfied. Also, for 2D electrostatic and 2D magnetostatic problems, $\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = 0$ and $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{0}$, respectively, are automatically satisfied.

The Linear Element Model

Linear Approximation Functions

The triangular Ω_e is defined by its three points, numbered counterclockwise; $\boldsymbol{P}_1: (x_1, y_1)$, $\boldsymbol{P}_2: (x_2, y_2)$, $\boldsymbol{P}_3: (x_3, y_3)$. (1)

Over Ω_e, $\psi(x, y) \approx c_1 + c_2 x + c_3 y$ (2)

(1), (2) →

$$\psi_1 \equiv \psi(\boldsymbol{P}_1) = c_1 + c_2 x_1 + c_3 y_1 \quad (3\text{-a})$$

$$\psi_2 \equiv \psi(\boldsymbol{P}_2) = c_1 + c_2 x_2 + c_3 y_2 \quad (3\text{-b})$$

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$$\begin{aligned} \psi_3 &\equiv \psi(\boldsymbol{P}_3) \\ &= c_1 + c_2 x_3 + c_3 y_3 \end{aligned} \tag{3-c}$$

(3) → $\boldsymbol{\psi} = \boldsymbol{A}\boldsymbol{c}$ in which $\boldsymbol{\psi} \equiv \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$, $\boldsymbol{A} \equiv \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$, $\boldsymbol{c} \equiv \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ (4)

(4) → $\boldsymbol{c} = \boldsymbol{A}^{-1}\boldsymbol{\psi}$ in which $A^{-1} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (5)

$$\psi(x, y) \approx \sum_{j=1}^{3} \psi_j \phi_j(x, y) \tag{6-a}$$

(2), (5) →

$$\phi_1(x, y) \equiv a_{11} + a_{21} x + a_{31} y \tag{6-b}$$
$$\phi_2(x, y) \equiv a_{12} + a_{22} x + a_{32} y \tag{6-c}$$
$$\phi_3(x, y) \equiv a_{13} + a_{23} x + a_{33} y \tag{6-d}$$

The Weak Form of the PDE

$$\iint_{\Omega_e} w \left[\frac{\partial}{\partial x}\left(\alpha \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha \frac{\partial \psi}{\partial y}\right) + \beta\psi - f \right] dxdy = 0 \tag{7}$$

The Chain Rule

$$\frac{\partial}{\partial u}\left[w\left(\alpha \frac{\partial \psi}{\partial u}\right)\right] = \frac{\partial w}{\partial u}\left(\alpha \frac{\partial \psi}{\partial u}\right) + w \frac{\partial}{\partial u}\left(\alpha \frac{\partial \psi}{\partial u}\right) \tag{CR}$$

The Stokes Theorem

$$\iint_{\Omega_e} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_{\partial\Omega_e} \boldsymbol{F} \cdot d\boldsymbol{l} \tag{ST}$$

(7), CR →

$$\begin{aligned} &- \iint_{\Omega_e} \frac{\partial w}{\partial x}\left(\alpha \frac{\partial \psi}{\partial x}\right) + \frac{\partial w}{\partial y}\left(\alpha \frac{\partial \psi}{\partial y}\right) dxdy + \beta \iint_{\Omega_e} w\psi \, dxdy \\ = \quad &- \iint_{\Omega_e} \frac{\partial}{\partial x}\left[w\left(\alpha \frac{\partial \psi}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[w\left(\alpha \frac{\partial \psi}{\partial y}\right)\right] dxdy + \iint_{\Omega_e} wf \, dxdy \end{aligned} \tag{8}$$

By defining $\boldsymbol{F} \equiv F_x \, \boldsymbol{a}_x + F_y \, \boldsymbol{a}_y$ in which $F_x \equiv -w\left(\alpha \frac{\partial \psi}{\partial y}\right)$ and $F_y \equiv w\left(\alpha \frac{\partial \psi}{\partial x}\right)$, the first expression in the right hand side of (8) can be reformulated as: $-\iint_{\Omega_e} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$. (9)

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In (ST), $d\boldsymbol{l}$ represents the differential length vector which is tangential to $\partial\Omega_e$, i.e., the boundary of Ω_e. $d\boldsymbol{l}$ can be expressed as $d\boldsymbol{l} = dl\,\boldsymbol{a}_l$ in which dl is its magnitude and $\boldsymbol{a}_l$ is a unit vector representing its direction. Noting that $\boldsymbol{n} \times \boldsymbol{a}_l = \boldsymbol{a}_z$, in which $\boldsymbol{n}$ represents the unit vector normal to $\partial\Omega_e$, the x and y components of $\boldsymbol{n}$ and $\boldsymbol{a}_l$ must satisfy $n_x a_{ly} - n_y a_{lx} = 1$. To express $\boldsymbol{a}_l$ in terms of $\boldsymbol{n}$, the above condition yields $a_{lx} = -n_y$ and $a_{ly} = n_x$. (10)

$$-\alpha \iint_{\Omega_e} \frac{\partial w}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \psi}{\partial y}\, dxdy + \beta \iint_{\Omega_e} w\psi\, dxdy$$

(8), (9), (10) → (11)

$$= \quad -\alpha \oint_{\partial\Omega_e} w\left[\frac{\partial \psi}{\partial x}n_x + \frac{\partial \psi}{\partial y}n_y\right] dl + \iint_{\Omega_e} wf\, dxdy$$

$$-\alpha \iint_{\Omega_e} \frac{\partial w}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \psi}{\partial y}\, dxdy + \beta \iint_{\Omega_e} w\psi\, dxdy$$

(11) → (12)

$$= \quad -\alpha \oint_{\partial\Omega_e} w(\boldsymbol{\nabla}\psi \cdot \boldsymbol{n})\, dl + \iint_{\Omega_e} wf\, dxdy$$

$$w(x,y) \in \{\phi_1(x,y), \phi_2(x,y), \phi_3(x,y)\} \quad (13)$$

For the k^{th} weighting function:

$$\sum_{j=1}^{3}\left(-\alpha \iint_{\Omega_e} \frac{\partial \phi_k}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_k}{\partial y}\frac{\partial \phi_j}{\partial y}\, dxdy\right)\psi_j$$

(6-a), (12), (13) →

$$+ \sum_{j=1}^{3}\left(\beta \iint_{\Omega_e} \phi_k\phi_j\, dxdy\right)\psi_j \quad (14)$$

$$= \quad -\alpha \oint_{\partial\Omega_e} \phi_k(\boldsymbol{\nabla}\psi \cdot \boldsymbol{n})\, dl + \iint_{\Omega_e} \phi_k f\, dxdy$$

$$M_{kj} \equiv -\alpha \iint_{\Omega_e} \frac{\partial \phi_k}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_k}{\partial y}\frac{\partial \phi_j}{\partial y}\, dxdy \quad (15\text{-a})$$

(14) → $T_{kj} \equiv \beta \iint_{\Omega_e} \phi_k\phi_j\, dxdy$ (15-b)

$$Q_k \equiv -\alpha \oint_{\partial\Omega_e} \phi_k(\boldsymbol{\nabla}\psi \cdot \boldsymbol{n})\, dl \quad (15\text{-c})$$

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$$f_k \equiv \iint_{\Omega_e} \phi_k f \, dxdy \tag{15-d}$$

Computation of the Element Model on the Master Triangle

Linear Approximation Functions on the Master Triangle

The master triangle on the ζ-η plane is defined by the following three points: $\boldsymbol{P}_1$: (0,0), $\boldsymbol{P}_2$: (1,0), $\boldsymbol{P}_3$: (0,1). (16)

(4), (16) → $$\boldsymbol{A} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{17}$$

(5), (17) → $$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{18}$$

(6), (18) →
$$\phi_1(\zeta, \eta) \equiv 1 - \zeta - \eta \tag{19-a}$$
$$\phi_2(\zeta, \eta) \equiv \zeta \tag{19-b}$$
$$\phi_3(\zeta, \eta) \equiv \eta \tag{19-c}$$

Isoparametric Elements

For Isoparametric elements, the same shape functions used to interpolate the primary unknown quantity inside an element are also used to interpolate the space coordinates x and y, i.e., $x = \sum_{j=1}^{3} x_j \phi_j(\zeta, \eta)$ and $y = \sum_{j=1}^{3} y_j \phi_j(\zeta, \eta)$. (20)

(1), (19), (20) →
$$x = x_1 + (x_2 - x_1)\zeta + (x_3 - x_1)\eta \tag{21-a}$$
$$y = y_1 + (y_2 - y_1)\zeta + (y_3 - x_1)\eta \tag{21-b}$$

The Chain Rule

$$\frac{\partial \phi}{\partial \zeta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \zeta} \tag{CR-1}$$
$$\frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta} \tag{CR-2}$$

(21), (CR-1), (CR-2) → $$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \frac{\partial \phi}{\partial \zeta} \\ \frac{\partial \phi}{\partial \zeta} \end{bmatrix}, \boldsymbol{J} \equiv \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_2 - y_1 \end{bmatrix} \tag{22}$$

(22) → $$\boldsymbol{J}^{-1} = \frac{1}{2S}\begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix}, S \equiv \frac{1}{2}\det(\boldsymbol{J}) \tag{23}$$

→ $$\frac{\partial \phi_1}{\partial x} = \frac{1}{2S}(y_2 - y_3) \tag{24-a}$$

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		$\frac{\partial \phi_1}{\partial y} = \frac{1}{2S}(x_3 - x_2)$	(24-b)
		$\frac{\partial \phi_2}{\partial x} = \frac{1}{2S}(y_3 - y_1)$	(24-c)
(19), (22), (23)		$\frac{\partial \phi_2}{\partial y} = \frac{1}{2S}(x_1 - x_3)$	(24-d)
		$\frac{\partial \phi_3}{\partial x} = \frac{1}{2S}(y_1 - y_2)$	(24-e)
		$\frac{\partial \phi_3}{\partial y} = \frac{1}{2S}(x_2 - x_1)$	(24-f)
		$M_{kj} = -\alpha \int_0^1 \int_0^{1-\eta} \left(\frac{\partial \phi_k}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_k}{\partial y}\frac{\partial \phi_j}{\partial y} \right) \det(\boldsymbol{J})\, d\zeta d\eta$	
(15), (24)	→	$T_{kj} = \beta \int_0^1 \int_0^{1-\eta} \left(\phi_k \phi_j\right) \det(\boldsymbol{J})\, d\zeta d\eta$	
		$f_k = f_e \int_0^1 \int_0^{1-\eta} \left(\phi_k\right) \det(\boldsymbol{J})\, d\zeta d\eta$	

Table 30. Derivation of the 1D Linear Element Model

The Linear Element Model

Linear Approximation Functions

		$\Omega_e \equiv [x_1, x_2]$	(1)
		Over Ω_e, $\psi(x) \approx c_1 + c_2 x$	(2)
(1), (2)	→	$\psi_1 \equiv \psi(x_1) = c_1 + c_2 x_1$	(3-a)
		$\psi_2 \equiv \psi(x_2) = c_1 + c_2 x_2$	(3-b)
(3)	→	$\boldsymbol{\psi} = \boldsymbol{A}\boldsymbol{c}$ in which $\boldsymbol{\psi} \equiv \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}$, $\boldsymbol{A} \equiv \begin{bmatrix}1 & x_1 \\ 1 & x_2\end{bmatrix}$, $\boldsymbol{c} \equiv \begin{bmatrix}c_1 \\ c_2\end{bmatrix}$	(4)
(4)	→	$\boldsymbol{c} = \boldsymbol{A}^{-1}\boldsymbol{\psi}$ in which $A^{-1} \equiv \begin{bmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$	(5)
(2), (5)	→	$\psi(x) \approx \sum_{j=1}^{2} \psi_j \phi_j(x)$	(6-a)

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$$\phi_1(x) \equiv a_{11} + a_{21}x \tag{6-b}$$
$$\phi_2(x) \equiv a_{12} + a_{22}x \tag{6-c}$$

The Weak Form of the PDE

$$\int_{\Omega_e} w\left[\frac{\partial}{\partial x}\left(\alpha\frac{\partial\psi}{\partial x}\right) + \beta\psi - f\right] dx = 0 \tag{7}$$

The Chain Rule

$$\frac{\partial}{\partial u}\left[w\left(\alpha\frac{\partial\psi}{\partial u}\right)\right] = \frac{\partial w}{\partial u}\left(\alpha\frac{\partial\psi}{\partial u}\right) + w\frac{\partial}{\partial u}\left(\alpha\frac{\partial\psi}{\partial u}\right) \quad \text{CR}$$

(7), CR →

$$-\alpha\int_{\Omega_e}\frac{\partial w}{\partial x}\frac{\partial\psi}{\partial x}\,dx + \beta\int_{\Omega_e} w\psi\,dx = -\alpha\int_{\Omega_e}\frac{\partial}{\partial x}\left[w\frac{\partial\psi}{\partial x}\right]dx + \int_{\Omega_e} wf\,dx \tag{8}$$

(8) →

$$-\alpha\int_{\Omega_e}\frac{\partial w}{\partial x}\frac{\partial\psi}{\partial x}\,dx + \beta\int_{\Omega_e} w\psi\,dx = -\alpha\left(w\frac{\partial\psi}{\partial x}(x=x_2) - w\frac{\partial\psi}{\partial x}(x=x_1)\right) + \int_{\Omega_e} wf\,dx \tag{9}$$

$$w(x,y) \in \{\phi_1(x,y), \phi_2(x,y), \phi_3(x,y)\} \tag{10}$$

For the k^{th} weighting function:

(6-a), (9), (10) →

$$\sum_{j=1}^{2}\left(-\alpha\int_{\Omega_e}\frac{\partial\phi_k}{\partial x}\frac{\partial\phi_j}{\partial x}\,dx\right)\psi_j + \sum_{j=1}^{2}\left(\beta\int_{\Omega_e}\phi_k\phi_j\,dx\right)\psi_j = -\alpha\left(\phi_k\frac{\partial\psi}{\partial x}(x=x_2) - \phi_k\frac{\partial\psi}{\partial x}(x=x_1)\right) + \int_{\Omega_e}\phi_k f\,dx \tag{11}$$

(11) →

$$M_{kj} \equiv -\alpha\int_{\Omega_e}\frac{\partial\phi_k}{\partial x}\frac{\partial\phi_j}{\partial x}\,dx$$

$$T_{kj} \equiv \beta\int_{\Omega_e}\phi_k\phi_j\,dx$$

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$$Q_k \equiv -\alpha\left(\phi_k \frac{\partial \psi}{\partial x}(x = x_2) - \phi_k \frac{\partial \psi}{\partial x}(x = x_1)\right)$$

$$f_k \equiv \int_{\Omega_e} \phi_k f \, dx$$

Table 31. Derivation of The Analytical Solution for the TM^z Mode of a 2D Cavity

TM^z Mode

$$\boldsymbol{A} = A_z \, \boldsymbol{a}_z \quad (1\text{-a})$$

$$\boldsymbol{F} = \boldsymbol{0} \quad (1\text{-b})$$

$$\boldsymbol{E} = -j\omega \boldsymbol{A} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) \quad (1\text{-c})$$

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A} \quad (1\text{-d})$$

$$\nabla^2 A_z + k^2 A_z = 0 \quad (1\text{-e})$$

2D Solutions

To obtain 2D solutions from the general 3D equations, all variations with respect to the third dimension, i.e., z, must be set to zero; In other words, $\frac{\partial}{\partial z} = 0$. (2)

Rectangular Cavity

$$E_x = 0 \quad (3\text{-a})$$

$$E_y = 0 \quad (3\text{-b})$$

$$E_z = -j\omega A_z \quad (3\text{-c})$$

$$H_x = -\frac{1}{\mu}\frac{\partial A_z}{\partial y} \quad (3\text{-d})$$

(1), (2) $\rightarrow$

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \quad (3\text{-e})$$

$$H_z = 0 \quad (3\text{-f})$$

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + k^2 A_z = 0 \quad (3\text{-g})$$

$$k_x^2 + k_y^2 = k^2 \quad (3\text{-h})$$

General Solution

$$A_z(x, y) = (C_1 \cos(k_x x) + D_1 \sin(k_x x))(C_1 \cos(k_x x) + D_1 \sin(k_x x)) \quad (4)$$

Boundary Conditions

$$E_z(x = 0) = 0 \quad (5\text{-a})$$

$$E_z(y = 0) = 0 \quad (5\text{-b})$$

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$$E_z(x=a)=0 \quad (5\text{-c})$$
$$E_z(y=b)=0 \quad (5\text{-d})$$

(4), (5-a) → $C_1=0$ (6-a)

(4), (5-b) → $C_2=0$ (6-b)

(4), (5-c) → $k_x a=m\pi,\ m=1,2,3,\ldots$ (6-c)

(4), (5-d) → $k_y b=n\pi,\ n=1,2,3,\ldots$ (6-d)

(4), (6) → $$A_z(x,y)=C\sin\left(m\frac{\pi}{a}x\right)\sin\left(n\frac{\pi}{b}y\right)$$

(3-h), (6-c), (6-d) → $$f_{mn}^r=\frac{c}{2a\sqrt{\mu_r\varepsilon_r}}\sqrt{m^2+\left(n\frac{a}{b}\right)^2}$$

Circular Cavity

$$E_\rho=0 \quad (7\text{-a})$$
$$E_\phi=0 \quad (7\text{-b})$$
$$E_z=-j\omega A_z \quad (7\text{-c})$$
$$H_\rho=\frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} \quad (7\text{-d})$$

(1), (2) → $$H_\phi=-\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} \quad (7\text{-e})$$

$$H_z=0 \quad (7\text{-f})$$
$$\frac{\partial^2 A_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial A_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2}+k^2A_z=0 \quad (7\text{-g})$$
$$k_\rho^2=k^2 \quad (7\text{-h})$$

General Solution

$$A_z(\rho,\phi)=\left(C_1J_m(k_\rho\rho)+D_1Y_m(k_\rho\rho)\right)(C_2\cos(m\phi)+D_2\sin(m\phi)) \quad (8)$$

Boundary Conditions

Fields must be finite everywhere. (9-a)

Fields must repeat every 2π radians in ϕ. (9-b)

$$E_z(\rho=a,\phi)=0 \quad (9\text{-c})$$

(8), (9-a) → $D_1=0$ (10-a)

To set the reference angle for rotation at $\phi=0$, set $D_2=0$. (10-b)

(8), (9-c) → $k_\rho a=\chi_{mn}$ in which χ_{mn} denotes the n^{th} root of the m^{th} Bessel function of the first kind. (10-c)

(8), (10) → $$A_z(\rho,\phi)=CJ_m\left(\frac{\chi_{mn}}{a}\rho\right)\cos(m\phi)$$

(7-h), (10-c) → $$f_{mn}^r=\frac{c}{2\pi a\sqrt{\mu_r\varepsilon_r}}\chi_{mn}$$

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12- Finite Difference Time Domain (FDTD)

Table 32. Derivation of the 2D TE^z Update Equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=0 \tag{1-a}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \tag{1-b}$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\mu\frac{\partial\boldsymbol{H}}{\partial t}-\boldsymbol{M} \tag{1-c}$$
$$\boldsymbol{\nabla}\times\boldsymbol{H}=\varepsilon\frac{\partial\boldsymbol{E}}{\partial t}+\boldsymbol{J} \tag{1-d}$$

$$\boldsymbol{M}\equiv\sigma^m\boldsymbol{H}+\boldsymbol{M}_i \tag{2-a}$$
$$\boldsymbol{J}\equiv\sigma^e\boldsymbol{E}+\boldsymbol{J}_i \tag{2-b}$$

(1-c), (2-a) $\rightarrow$
$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\mu\frac{\partial\boldsymbol{H}}{\partial t}-\sigma^m\boldsymbol{H}-\boldsymbol{M}_i \tag{3-a}$$
(1-d), (2-b) $\rightarrow$
$$\boldsymbol{\nabla}\times\boldsymbol{H}=\varepsilon\frac{\partial\boldsymbol{E}}{\partial t}+\sigma^e\boldsymbol{E}+\boldsymbol{J}_i \tag{3-b}$$

(3) $\rightarrow$
$$\frac{\partial\boldsymbol{H}}{\partial t}=-\frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{E}-\frac{\sigma^m}{\mu}\boldsymbol{H}-\frac{1}{\mu}\boldsymbol{M}_i \tag{4-a}$$
$$\frac{\partial\boldsymbol{E}}{\partial t}=\frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{H}-\frac{\sigma^e}{\varepsilon}\boldsymbol{E}-\frac{1}{\varepsilon}\boldsymbol{J}_i \tag{4-b}$$

(4-a) $\rightarrow$
$$\frac{\partial H_x}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}\right)-\frac{\sigma^m}{\mu}H_x-\frac{1}{\mu}M_{ix} \tag{5-a}$$
$$\frac{\partial H_y}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)-\frac{\sigma^m}{\mu}H_y-\frac{1}{\mu}M_{iy} \tag{5-b}$$
$$\frac{\partial H_z}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)-\frac{\sigma^m}{\mu}H_z-\frac{1}{\mu}M_{iz} \tag{5-c}$$

(4-b) $\rightarrow$
$$\frac{\partial E_x}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)-\frac{\sigma^e}{\varepsilon}E_x-\frac{1}{\varepsilon}J_{ix} \tag{6-a}$$
$$\frac{\partial E_y}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)-\frac{\sigma^e}{\varepsilon}E_y-\frac{1}{\varepsilon}J_{iy} \tag{6-b}$$
$$\frac{\partial E_z}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)-\frac{\sigma^e}{\varepsilon}E_z-\frac{1}{\varepsilon}J_{iz} \tag{6-c}$$

2D TE^z Mode

$$E_z=0 \tag{7-a}$$
$$\frac{\partial}{\partial z}=0 \tag{7-b}$$

(5), (6), (7) $\rightarrow$
$$\frac{\partial H_z}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)-\frac{\sigma^m}{\mu}H_z-\frac{1}{\mu}M_{iz} \tag{8-a}$$

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$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\sigma^e}{\varepsilon}E_x - \frac{1}{\varepsilon}J_{ix} \tag{8-b}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma^e}{\varepsilon}E_y - \frac{1}{\varepsilon}J_{iy} \tag{8-c}$$

Discretization

Discretization of Space

The top-right corner of cell (i,j) is located at $(i\Delta x, j\Delta y)$ (9-a)

$H_z(i,j)$ is located at $\left(\left(i-\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right)$ (9-b)

(9-a) → $E_x(i,j)$ is located at $\left(\left(i-\frac{1}{2}\right)\Delta x, (j-1)\Delta y\right)$ (9-c)

$E_y(i,j)$ is located at $\left((i-1)\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right)$ (9-d)

Discretization of H_z

(8-a) → $$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) - \frac{\sigma^m}{\mu}H_z - \frac{1}{\mu}M_{iz} \tag{10}$$

Temporal Derivative of H_z

$$\frac{\partial H_z}{\partial t} \approx \frac{1}{\Delta t}\left(H_z^{n+\frac{1}{2}}(i,j) - H_z^{n-\frac{1}{2}}(i,j)\right) \tag{11}$$

Partial Derivative of E_x at the location of H_z

(9-b), (10) → $$\begin{aligned}\frac{\partial E_x}{\partial y} \approx\ & \frac{1}{\Delta y}E_x^n\left(\left(i-\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta y + \frac{1}{2}\Delta y\right) \\ -\ & \frac{1}{\Delta y}E_x^n\left(\left(i-\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta y - \frac{1}{2}\Delta y\right)\end{aligned} \tag{12}$$

(9-c), (12) → $$\begin{aligned}\frac{\partial E_x}{\partial y} \approx\ & \frac{1}{\Delta y}E_x^n\left(\left(i-\frac{1}{2}\right)\Delta x, ([j+1]-1)\Delta y\right) \\ -\ & \frac{1}{\Delta y}E_x^n\left(\left(i-\frac{1}{2}\right)\Delta x, (j-1)\Delta y\right)\end{aligned} \tag{13}$$

(13) → $$\frac{\partial E_x}{\partial y} \approx \frac{1}{\Delta y}\left(E_x^n(i,j+1) - E_x^n(i,j)\right) \tag{14}$$

Partial Derivative of E_y at the location of H_z

(9-b), (10) → $$\frac{\partial E_y}{\partial x} \approx \frac{1}{\Delta x}E_y^n\left(\left(i-\frac{1}{2}\right)\Delta x + \frac{1}{2}\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right) \tag{15}$$

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$$- \frac{1}{\Delta x} E_y^n \left(\left(i - \frac{1}{2}\right)\Delta x - \frac{1}{2}\Delta x, \left(j - \frac{1}{2}\right)\Delta y \right)$$

(9-d), (15) → $$\frac{\partial E_y}{\partial x} \approx \frac{1}{\Delta x} E_y^n \left(([i+1]-1)\Delta x + \frac{1}{2}\Delta x, \left(j - \frac{1}{2}\right)\Delta y \right) - \frac{1}{\Delta x} E_y^n \left((i-1)\Delta x, \left(j - \frac{1}{2}\right)\Delta y \right) \quad (16)$$

(16) → $$\frac{\partial E_y}{\partial x} \approx \frac{1}{\Delta x}\left(E_y^n(i+1,j) - E_y^n(i,j)\right) \quad (17)$$

Value of H_z

$$H_z \approx \frac{1}{2}\left(H_z^{n+\frac{1}{2}}(i,j) + H_z^{n-\frac{1}{2}}(i,j) \right) \quad (18)$$

(10), (11), (14), (17), (18) →
$$H_z^{n+\frac{1}{2}}(i,j) = \frac{2\mu - \Delta t\sigma^m}{2\mu + \Delta t\sigma^m} H_z^{n-\frac{1}{2}}(i,j) + \frac{2\Delta t}{(2\mu + \Delta t\sigma^m)\Delta y}\left(E_x^n(i,j+1) - E_x^n(i,j)\right) - \frac{2\Delta t}{(2\mu + \Delta t\sigma^m)\Delta x}\left(E_y^n(i+1,j) - E_y^n(i,j)\right) - \frac{2\Delta t}{2\mu + \Delta t\sigma^m} M_{iz}^n(i,j)$$

Discretization of E_x

(8-b) → $$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\sigma^e}{\varepsilon} E_x - \frac{1}{\varepsilon} J_{ix} \quad (19)$$

Temporal Derivative of E_x

$$\frac{\partial E_x}{\partial t} \approx \frac{1}{\Delta t}\left(E_x^{n+1}(i,j) - E_x^n(i,j)\right) \quad (20)$$

Partial Derivative of H_z at the location of E_x

(9-c), (19) → $$\frac{\partial H_z}{\partial y} \approx \frac{1}{\Delta y} H_z^{n+\frac{1}{2}}\left(\left(i - \frac{1}{2}\right)\Delta x, (j-1)\Delta y + \frac{1}{2}\Delta y \right) - \frac{1}{\Delta y} H_z^{n+\frac{1}{2}}\left(\left(i - \frac{1}{2}\right)\Delta x, (j-1)\Delta y - \frac{1}{2}\Delta y \right) \quad (21)$$

(9-b), (21) → $$\frac{\partial H_z}{\partial y} \approx \frac{1}{\Delta y} H_z^{n+\frac{1}{2}}\left(\left(i - \frac{1}{2}\right)\Delta x, \left(j - \frac{1}{2}\right)\Delta y \right) - \frac{1}{\Delta} H_z^{n+\frac{1}{2}}\left(\left(i - \frac{1}{2}\right)\Delta x, \left([j-1] - \frac{1}{2}\right)\Delta y \right) \quad (22)$$

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(22) → $$\frac{\partial H_z}{\partial y} \approx \frac{1}{\Delta y}\left(H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i,j-1)\right) \quad (23)$$

Value of E_x

$$E_x \approx \frac{1}{2}\left(E_x^{n+1}(i,j) + E_x^{n}(i,j)\right) \quad (23)$$

(19), (20), (23), (24) →
$$\begin{aligned} E_x^{n+1}(i,j) &= \frac{2\varepsilon - \Delta t\sigma^e}{2\varepsilon + \Delta t\sigma^e} E_x^n(i,j) \\ &+ \frac{2\Delta t}{(2\varepsilon + \Delta t\sigma^e)\Delta y}\left(H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i,j-1)\right) \\ &- \frac{2\Delta t}{2\varepsilon + \Delta t\sigma^e} J_{ix}^{n+\frac{1}{2}}(i,j) \end{aligned}$$

Discretization of E_y

(8-c) → $$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma^e}{\varepsilon}E_y - \frac{1}{\varepsilon}J_{iy} \quad (25)$$

Temporal Derivative of E_y

$$\frac{\partial E_y}{\partial t} \approx \frac{1}{\Delta t}\left(E_y^{n+1}(i,j) - E_y^{n}(i,j)\right) \quad (26)$$

Partial Derivative of H_z at the location of E_y

(9-d), (25) →
$$\begin{aligned} \frac{\partial H_z}{\partial x} &\approx \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left((i-1)\Delta x + \frac{1}{2}\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right) \\ &- \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left((i-1)\Delta x - \frac{1}{2}\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right) \end{aligned} \quad (27)$$

(9-b), (27) →
$$\begin{aligned} \frac{\partial H_z}{\partial x} &\approx \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left(\left(i-\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right) \\ &- \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left(\left([i-1]-\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta y\right) \end{aligned} \quad (28)$$

(28) → $$\frac{\partial H_z}{\partial x} \approx \frac{1}{\Delta x}\left(H_z^{n+\frac{1}{2}}(i,j) - H_z^{n+\frac{1}{2}}(i-1,j)\right) \quad (29)$$

Value of E_y

$$E_y \approx \frac{1}{2}\left(E_y^{n+1}(i,j) + E_y^{n}(i,j)\right) \quad (30)$$

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$$
\begin{array}{lcl}
E_y^{n+1}(i,j) & = & \dfrac{2\varepsilon-\Delta t\sigma^e}{2\varepsilon+\Delta t\sigma^e}E_y^n(i,j) \\
\text{(25), (26), (29), (30)} \quad \rightarrow & - & \dfrac{2\Delta t}{(2\varepsilon+\Delta t\sigma^e)\Delta x}\left(H_z^{n+\frac{1}{2}}(i,j)-H_z^{n+\frac{1}{2}}(i-1,j)\right) \\
& - & \dfrac{2\Delta t}{2\varepsilon+\Delta t\sigma^e}J_{iy}^{n+\frac{1}{2}}(i,j)
\end{array}
$$

Table 33. Derivation of the Split-PML Absorbing Boundary Condition

2D TE^z Mode in Source-Free Region: Maxwell's Equations

$$\frac{\partial H_z}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)-\frac{\sigma^m}{\mu}H_z \tag{1-a}$$

$$\frac{\partial E_x}{\partial t}=\frac{1}{\varepsilon}\frac{\partial H_z}{\partial y}-\frac{\sigma^e}{\varepsilon}E_x \tag{1-b}$$

$$\frac{\partial E_y}{\partial t}=-\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x}-\frac{\sigma^e}{\varepsilon}E_y \tag{1-c}$$

(1) $\rightarrow$

$$H_z=-j\frac{1}{\omega\mu S^m}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right) \qquad S^m\equiv 1-j\frac{\sigma^m}{\omega\mu} \tag{2-a}$$

$$E_x=-j\frac{1}{\omega\varepsilon S^e}\frac{\partial H_z}{\partial y} \qquad S^e\equiv 1-j\frac{\sigma^e}{\omega\varepsilon} \tag{2-b}$$

$$E_y=j\frac{1}{\omega\varepsilon S^e}\frac{\partial H_z}{\partial x} \tag{2-c}$$

Splitting of H_z and Introduction of Anisotropy in Conductivities

- $H_z=H_{zx}+H_{zy}$ (3)
- $(H_{zx},\sigma_x^m),(H_{zy},\sigma_y^m),(E_x,\sigma_y^e),(E_y,\sigma_x^e)$

(2), (3) $\rightarrow$

$$H_{zx}=j\frac{1}{\omega\mu S_x^m}\frac{\partial E_y}{\partial x} \qquad S_x^m\equiv 1-j\frac{\sigma_x^m}{\omega\mu} \tag{4-a}$$

$$H_{zy}=-j\frac{1}{\omega\mu S_y^m}\frac{\partial E_x}{\partial y} \qquad S_y^m\equiv 1-j\frac{\sigma_y^m}{\omega\mu} \tag{4-b}$$

$$E_x=-j\frac{1}{\omega\varepsilon S_y^e}\frac{\partial H_z}{\partial y} \qquad S_y^e\equiv 1-j\frac{\sigma_y^e}{\omega\varepsilon} \tag{4-c}$$

$$E_y=j\frac{1}{\omega\varepsilon S_x^e}\frac{\partial H_z}{\partial x} \qquad S_x^e\equiv 1-j\frac{\sigma_x^e}{\omega\varepsilon} \tag{4-d}$$

2D TE^z Mode in Source-Free Region: Solutions

$$\boldsymbol{E} \equiv \left(E_{0x}\boldsymbol{a}_x+E_{0y}\boldsymbol{a}_y\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \qquad E_{0x}^2+E_{0y}^2\equiv E_0^2 \tag{5-a}$$

$$\boldsymbol{H} \equiv (H_{0z}\boldsymbol{a}_z)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \tag{5-b}$$

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$$= \left(\frac{1}{Z}E_0\boldsymbol{a}_z\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$
$$= \left(H_{0zx}\boldsymbol{a}_z + H_{0zy}\boldsymbol{a}_z\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\boldsymbol{k} \equiv k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y \quad (5\text{-c})$$

(4-a), (5-a) → $$H_{0zx} = \frac{k_x}{\omega\mu S_x^m}E_{0y} \quad (6\text{-a})$$

(4-b), (5-a) → $$H_{0zy} = -\frac{k_y}{\omega\mu S_y^m}E_{0x} \quad (6\text{-b})$$

(4-c), (5-b) → $$E_{0x} = -\frac{k_y}{\omega\varepsilon S_y^e}H_{0z} \quad (6\text{-c})$$

(4-d), (5-d) → $$E_{0y} = \frac{k_x}{\omega\varepsilon S_x^e}H_{0z} \quad (6\text{-d})$$

(6-a), (6-d) → $$H_{0zx} = \frac{k_x^2}{\omega^2\mu\varepsilon S_x^m S_x^e}H_{0z} \quad (7\text{-a})$$

(6-b), (6-c) → $$H_{0zy} = \frac{k_y^2}{\omega^2\mu\varepsilon S_y^m S_y^e}H_{0z} \quad (7\text{-b})$$

(7) → $$\omega^2\mu\varepsilon = \left(\frac{k_x}{\sqrt{S_x^m S_x^e}}\right)^2 + \left(\frac{k_y}{\sqrt{S_y^m S_y^e}}\right)^2 \quad (8)$$

(8) →
$$k_x = \omega\sqrt{\mu\varepsilon}\sqrt{S_x^m S_x^e}\cos\theta \quad (9\text{-a})$$
$$k_y = \omega\sqrt{\mu\varepsilon}\sqrt{S_y^m S_y^e}\sin\theta \quad (9\text{-b})$$

(5-a), (5-b), (6-c), (6-d), (9) →
$$Z \equiv \frac{E_0}{H_{0z}} = \sqrt{\frac{\mu}{\varepsilon}}W \quad (10\text{-a})$$

$$W \equiv \sqrt{\left(\sqrt{\frac{S_x^m}{S_x^e}}\cos\theta\right)^2 + \left(\sqrt{\frac{S_y^m}{S_y^e}}\sin\theta\right)^2} \quad (10\text{-b})$$

2D TE^z Mode in Matched Medium

(6-c), (6-d), (9-a), (9-b), (10) →
$$\boldsymbol{E} = \frac{E_0}{W}\left(-\sqrt{\frac{S_y^m}{S_y^e}}\sin\theta\,\boldsymbol{a}_x + \sqrt{\frac{S_x^m}{S_x^e}}\cos\theta\,\boldsymbol{a}_y\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \quad (11\text{-a})$$

(5-b) → $$\boldsymbol{H} = \left(\frac{E_0}{Z}\boldsymbol{a}_z\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \quad (11\text{-b})$$

(9) → $$\boldsymbol{k} = \omega\sqrt{\mu\varepsilon}\left(\sqrt{S_x^m S_x^e}\cos\theta\,\boldsymbol{a}_x + \sqrt{S_y^m S_y^e}\sin\theta\,\boldsymbol{a}_y\right) \quad (11\text{-c})$$

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(10) → $Z = \sqrt{\frac{\mu}{\varepsilon}} W \qquad W \equiv \sqrt{\left(\sqrt{\frac{S_x^m}{S_x^e}}\cos\theta\right)^2 + \left(\sqrt{\frac{S_y^m}{S_y^e}}\sin\theta\right)^2}$ (11-d)

Matching Conditions

$$S_x^m = S_x^e \quad (12\text{-a})$$
$$S_y^m = S_y^e \quad (12\text{-b})$$

(4), (12) →

$$\sigma_x^m = \frac{\mu}{\varepsilon}\sigma_x^e \quad (13\text{-a})$$
$$\sigma_y^m = \frac{\mu}{\varepsilon}\sigma_y^e \quad (13\text{-b})$$
$$\boldsymbol{E} = E_0\left(-\sin\theta\,\boldsymbol{a}_x + \cos\theta\,\boldsymbol{a}_y\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \quad (13\text{-c})$$
$$\boldsymbol{H} = \left(\frac{E_0}{Z}\boldsymbol{a}_z\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \quad (13\text{-d})$$

(11), (12) →

$$\boldsymbol{k} = \omega\sqrt{\mu\varepsilon}\left(\cos\theta\,\boldsymbol{a}_x + \sin\theta\,\boldsymbol{a}_y\right) - j\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_x^e\cos\theta\,\boldsymbol{a}_x + \sigma_y^e\sin\theta\,\boldsymbol{a}_y\right) \quad (13\text{-e})$$
$$Z = \sqrt{\frac{\mu}{\varepsilon}} \quad (13\text{-f})$$

2D TE^z Mode in Matched Medium: Reflection & Transmission

Incidence Field

$$\boldsymbol{a}_p^i = \cos\theta_i\,\boldsymbol{a}_x + \sin\theta_i\,\boldsymbol{a}_y \quad (14\text{-a})$$
$$\boldsymbol{E}^i = E_0\left(-\sin\theta_i\,\boldsymbol{a}_x + \cos\theta_i\,\boldsymbol{a}_y\right)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x1}^e\cos\theta_i x + \sigma_{y1}^e\sin\theta_i y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^i\cdot\boldsymbol{r}} \quad (14\text{-b})$$
$$\boldsymbol{H}^i = \frac{1}{Z}(E_0\boldsymbol{a}_z)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x1}^e\cos\theta_i x + \sigma_{y1}^e\sin\theta_i y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^i\cdot\boldsymbol{r}} \quad (14\text{-c})$$

Reflected Field

$$\boldsymbol{a}_p^r = -\cos\theta_r\,\boldsymbol{a}_x + \sin\theta_r\,\boldsymbol{a}_y \quad (15\text{-a})$$
$$\boldsymbol{E}^r = \Gamma E_0\left(\sin\theta_r\,\boldsymbol{a}_x + \cos\theta_r\,\boldsymbol{a}_y\right)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x1}^e\cos\theta_r x + \sigma_{y1}^e\sin\theta_r y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^r\cdot\boldsymbol{r}} \quad (15\text{-b})$$
$$\boldsymbol{H}^r = \frac{1}{Z}(-\Gamma E_0\boldsymbol{a}_z)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x1}^e\cos\theta_r x + \sigma_{y1}^e\sin\theta_r y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^r\cdot\boldsymbol{r}} \quad (15\text{-c})$$

Transmitted Field

$$\boldsymbol{a}_p^t = \cos\theta_t\,\boldsymbol{a}_x + \sin\theta_t\,\boldsymbol{a}_y \quad (16\text{-a})$$
$$\boldsymbol{E}^t = TE_0\left(-\sin\theta_t\,\boldsymbol{a}_x + \cos\theta_t\,\boldsymbol{a}_y\right)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x2}^e\cos\theta_t x + \sigma_{y2}^e\sin\theta_t y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^t\cdot\boldsymbol{r}} \quad (16\text{-b})$$
$$\boldsymbol{H}^t = \frac{1}{Z}(TE_0\boldsymbol{a}_z)e^{-\sqrt{\frac{\mu}{\varepsilon}}\left(\sigma_{x2}^e\cos\theta_t x + \sigma_{y2}^e\sin\theta_t y\right)}e^{-j\omega\sqrt{\mu\varepsilon}\boldsymbol{a}_p^t\cdot\boldsymbol{r}} \quad (16\text{-c})$$

Along the Boundary Between the Two Media

$$\boldsymbol{r}_b = y\boldsymbol{a}_y \quad (17\text{-a})$$

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$$\boldsymbol{n} = \boldsymbol{a}_x \tag{17-b}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \qquad \boldsymbol{E}_2 \equiv \boldsymbol{E}^t \qquad \boldsymbol{E}_1 \equiv \boldsymbol{E}^i + \boldsymbol{E}^r \tag{18-a}$$

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0 \qquad \boldsymbol{H}_2 \equiv \boldsymbol{H}^t \qquad \boldsymbol{H}_1 \equiv \boldsymbol{H}^i + \boldsymbol{H}^r \tag{18-b}$$

(18) $\rightarrow$

$$E_y^i(\boldsymbol{r}_b) + E_y^r(\boldsymbol{r}_b) = E_y^t(\boldsymbol{r}_b) \tag{19-a}$$

$$H_z^i(\boldsymbol{r}_b) + H_z^r(\boldsymbol{r}_b) = H_z^t(\boldsymbol{r}_b) \tag{19-b}$$

(14), (15), (16), (19-a) $\rightarrow$

$$E_0 \cos\theta_i \, e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y1}^e \sin\theta_i y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_i y} + \Gamma E_0 \cos\theta_r \, e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y1}^e \sin\theta_r y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_r y} = T E_0 \cos\theta_t \, e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y2}^e \sin\theta_t y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_t y} \tag{20-a}$$

(14), (15), (16), (19-b) $\rightarrow$

$$\frac{E_0}{Z} e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y1}^e \sin\theta_i y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_i y} - \frac{\Gamma E_0}{Z} e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y1}^e \sin\theta_r y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_r y} = \frac{T E_0}{Z} e^{-\sqrt{\frac{\mu}{\varepsilon}}(\sigma_{y2}^e \sin\theta_t y)} e^{-j\omega\sqrt{\mu\varepsilon}\sin\theta_t y} \tag{20-b}$$

(20) $\rightarrow$

$\theta_t = \theta_r = \theta_i$

With Condition $\sigma_{y1}^e = \sigma_{y2}^e$

$1 + \Gamma = T$

$1 - \Gamma = T$

Table 34. Derivation of the TF/SF Consistency Equations

Assume a TF/SF boundary on the Yee grid. Any field components on or inside the boundary is considered a ***Total Field (TF)*** and any field component outside the boundary is considered ***Scattered Field (SF)***. Since $\boldsymbol{E}^{Total} = \boldsymbol{E}^{Scattered} + \boldsymbol{E}^{inc}$ and $\boldsymbol{H}^{Total} = \boldsymbol{H}^{Scattered} + \boldsymbol{H}^{inc}$, a correction term must be added to each update equation to enforce consistency.

Region O1

$$\begin{aligned} H_z^{n+\frac{1}{2}SF}(i,j) &= C_{hh}(i,j) H_z^{n-\frac{1}{2}SF}(i,j) \\ &+ C_{hex}(i,j)\left(E_x^{nTF}(i,j+1) - E_x^{nSF}(i,j)\right) \\ &+ C_{hey}(i,j)\left(E_y^{nSF}(i+1,j) - E_y^{nSF}(i,j)\right) \end{aligned}$$

Update equations for E_x and E_y require no corrections because all their terms are in the SF region.

Region O2

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$$H_z^{n+\frac{1}{2}^{SF}}(i,j) = C_{hh}(i,j)H_z^{n-\frac{1}{2}^{SF}}(i,j) + C_{hex}(i,j)\left(E_x^{n^{SF}}(i,j+1) - E_x^{n^{SF}}(i,j)\right) + C_{hey}(i,j)\left(E_y^{n^{SF}}(i+1,j) - E_y^{n^{TF}}(i,j)\right)$$

$$E_y^{n+1^{TF}}(i,j) = C_{eye}(i,j)E_y^{n^{TF}}(i,j) + C_{eyh}(i,j)\left(H_z^{n+\frac{1}{2}^{SF}}(i,j) - H_z^{n+\frac{1}{2}^{TF}}(i-1,j)\right)$$

Update equation for E_x requires no corrections because all its terms are in the SF region.

Region O3

$$H_z^{n+\frac{1}{2}^{SF}}(i,j) = C_{hh}(i,j)H_z^{n-\frac{1}{2}^{SF}}(i,j) + C_{hex}(i,j)\left(E_x^{n^{SF}}(i,j+1) - E_x^{n^{TF}}(i,j)\right) + C_{hey}(i,j)\left(E_y^{n^{SF}}(i+1,j) - E_y^{n^{SF}}(i,j)\right)$$

$$E_x^{n+1^{TF}}(i,j) = C_{exe}(i,j)E_x^{n^{TF}}(i,j) + C_{exh}(i,j)\left(H_z^{n+\frac{1}{2}^{SF}}(i,j) - H_z^{n+\frac{1}{2}^{TF}}(i,j-1)\right)$$

Update equation for E_y requires no corrections because all its terms are in the SF region.

Region O4

$$H_z^{n+\frac{1}{2}^{SF}}(i,j) = C_{hh}(i,j)H_z^{n-\frac{1}{2}^{SF}}(i,j) + C_{hex}(i,j)\left(E_x^{n^{SF}}(i,j+1) - E_x^{n^{SF}}(i,j)\right) + C_{hey}(i,j)\left(E_y^{n^{TF}}(i+1,j) - E_y^{n^{SF}}(i,j)\right)$$

Update equations for E_x and E_y require no corrections because all their terms are in the SF region.

Region I1

$$E_x^{n+1^{TF}}(i,j) = C_{exe}(i,j)E_x^{n^{TF}}(i,j) + C_{exh}(i,j)\left(H_z^{n+\frac{1}{2}^{TF}}(i,j) - H_z^{n+\frac{1}{2}^{SF}}(i,j-1)\right)$$

Update equations for H_z and E_y require no corrections because all their terms are in the TF region.

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Region I2

Update equations for H_z, E_x and E_y require no corrections because all their terms are in the TF region.

Region *I*3

Update equations for H_z, E_x and E_y require no corrections because all their terms are in the TF region.

Region I4

$$E_y^{n+1^{TF}}(i,j) = C_{eye}(i,j)E_y^{n^{TF}}(i,j) + C_{eyh}(i,j)\left(H_z^{n+\frac{1}{2}^{TF}}(i,j) - H_z^{n+\frac{1}{2}^{SF}}(i-1,j)\right)$$

Update equations for H_z and E_x require no corrections because all their terms are in the TF region.

Table 35. Derivation of the 1D TE^z Update Equations

2D TE^z Mode: Maxwell's Equations

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) - \frac{\sigma^m}{\mu}H_z - \frac{1}{\mu}M_{iz} \quad (1\text{-a})$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\sigma^e}{\varepsilon}E_x - \frac{1}{\varepsilon}J_{ix} \quad (1\text{-b})$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma^e}{\varepsilon}E_y - \frac{1}{\varepsilon}J_{iy} \quad (1\text{-c})$$

$$\frac{\partial}{\partial y} = 0 \quad (2)$$

1D TE^z Mode: Maxwell's Equations

(1), (2) →

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\frac{\partial E_y}{\partial x} - \frac{\sigma^m}{\mu}H_z - \frac{1}{\mu}M_{iz} \quad (3\text{-a})$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma^e}{\varepsilon}E_y - \frac{1}{\varepsilon}J_{iy} \quad (3\text{-b})$$

Discretization

Discretization of Space

The right corner of the i^{th} cell is located at $(i\Delta x)$ (4-a)

(4-a) → $H_z(i)$ is located at $\left(i - \frac{1}{2}\right)\Delta x$ (4-b)

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$E_y(i)$ is located at $(i-1)\Delta x$ (4-c)

Discretization of H_z

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\frac{\partial E_y}{\partial x} - \frac{\sigma^m}{\mu}H_z - \frac{1}{\mu}M_{iz} \tag{5}$$

Temporal Derivative of H_z

$$\frac{\partial H_z}{\partial t} \approx \frac{1}{\Delta t}\left(H_z^{n+\frac{1}{2}}(i) - H_z^{n-\frac{1}{2}}(i)\right) \tag{6}$$

Partial Derivative of E_y at the location of H_z

(4-b), (5) →
$$\begin{aligned}\frac{\partial E_y}{\partial x} &\approx \frac{1}{\Delta x}E_y^n\left(\left(i-\frac{1}{2}\right)\Delta x + \frac{1}{2}\Delta x\right)\\ &- \frac{1}{\Delta x}E_y^n\left(\left(i-\frac{1}{2}\right)\Delta x - \frac{1}{2}\Delta x\right)\end{aligned} \tag{7}$$

(4-c), (7) →
$$\begin{aligned}\frac{\partial E_y}{\partial x} &\approx \frac{1}{\Delta x}E_y^n\big(([i+1]-1)\Delta x\big)\\ &- \frac{1}{\Delta x}E_y^n\big((i-1)\Delta x\big)\end{aligned} \tag{8}$$

(8) →
$$\frac{\partial E_y}{\partial x} \approx \frac{1}{\Delta x}E_y^n(i+1) - \frac{1}{\Delta x}E_y^n(i) \tag{9}$$

Value of H_z

$$H_z \approx \frac{1}{2}\left(H_z^{n+\frac{1}{2}}(i) + H_z^{n-\frac{1}{2}}(i)\right) \tag{10}$$

(5), (6), (9), (10) →
$$\begin{aligned}H_z^{n+\frac{1}{2}}(i) &= \frac{2\mu - \Delta t\sigma^m}{2\mu + \Delta t\sigma^m}H_z^{n-\frac{1}{2}}(i)\\ &- \frac{2\Delta t}{(2\mu + \Delta t\sigma^m)\Delta x}\left(E_y^n(i+1) - E_y^n(i)\right)\\ &- \frac{2\Delta t}{2\mu + \Delta t\sigma^m}M_{iz}^n(i)\end{aligned}$$

Discretization of E_y

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma^e}{\varepsilon}E_y - \frac{1}{\varepsilon}J_{iy} \tag{11}$$

Temporal Derivative of E_y

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$$\frac{\partial E_y}{\partial t} \approx \frac{1}{\Delta t}\left(E_y^{n+1}(i) - E_y^n(i)\right) \tag{12}$$

Partial Derivative of H_z at the location of E_y

$$\text{(4-b), (11)} \rightarrow \quad \begin{aligned} \frac{\partial H_z}{\partial x} &\approx \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left((i-1)\Delta x + \frac{1}{2}\Delta x\right) \\ &- \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left((i-1)\Delta x - \frac{1}{2}\Delta x\right) \end{aligned} \tag{13}$$

$$\text{(4-c), (13)} \rightarrow \quad \begin{aligned} \frac{\partial H_z}{\partial x} &\approx \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left(\left(i - \frac{1}{2}\right)\Delta x\right) \\ &- \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}\left(\left([i-1] - \frac{1}{2}\right)\Delta x\right) \end{aligned} \tag{14}$$

$$\text{(14)} \rightarrow \quad \frac{\partial H_z}{\partial x} \approx \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}(i) - \frac{1}{\Delta x} H_z^{n+\frac{1}{2}}(i-1) \tag{15}$$

Value of E_y

$$E_y \approx \frac{1}{2}\left(E_y^{n+1}(i) + E_y^n(i)\right) \tag{16}$$

$$\text{(11), (12), (15), (16)} \rightarrow \quad \begin{aligned} E_y^{n+1}(i) &= \frac{2\varepsilon - \Delta t\sigma^e}{2\varepsilon + \Delta t\sigma^e} E_y^n(i) \\ &- \frac{2\Delta t}{(2\varepsilon + \Delta t\sigma^e)\Delta x}\left(H_z^{n+\frac{1}{2}}(i) - H_z^{n+\frac{1}{2}}(i-1)\right) \\ &- \frac{2\Delta t}{2\varepsilon + \Delta t\sigma^e} J_{iy}^{n+\frac{1}{2}}(i) \end{aligned}$$

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13- Method of Moments (MoM)

Table 36. Derivation of the EFIE and MFIE for Scattering from PEC Objects

General EFIE and MFIE for Scattering from PEC Objects

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{F} \tag{1-a}$$

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A} - j\omega\boldsymbol{F} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) \tag{1-b}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \mu\iint_S \boldsymbol{J}_s(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')dS' \tag{2-a}$$

$$\boldsymbol{F}(\boldsymbol{r}) = \varepsilon\iint_S \boldsymbol{M}_s(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')dS' \tag{2-b}$$

$$G(\boldsymbol{r},\boldsymbol{r}') \equiv \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{2-c}$$

$$\boldsymbol{\nabla}\times(\psi\boldsymbol{A}) = \psi\boldsymbol{\nabla}\times\boldsymbol{A} + (\boldsymbol{\nabla}\psi)\times\boldsymbol{A} \tag{VI-1}$$

Physical Equivalence Theorem

The imaginary surface S that coincides with the boundary of the PEC object is defined by $\boldsymbol{r} = \boldsymbol{r}_b$. (3-a)

$$\boldsymbol{M}_s(\boldsymbol{r}=\boldsymbol{r}_b) = \boldsymbol{0} \tag{3-b}$$

$$\boldsymbol{n}\times\boldsymbol{E}_S(\boldsymbol{r}=\boldsymbol{r}_b) = -\boldsymbol{n}\times\boldsymbol{E}_{inc}(\boldsymbol{r}=\boldsymbol{r}_b) \tag{3-c}$$

$$\boldsymbol{n}\times\boldsymbol{H}_S(\boldsymbol{r}=\boldsymbol{r}_b) - \boldsymbol{J}_s(\boldsymbol{r}=\boldsymbol{r}_b) = -\boldsymbol{n}\times\boldsymbol{H}_{inc}(\boldsymbol{r}=\boldsymbol{r}_b) \tag{3-d}$$

(3-b) → $$\boldsymbol{F} = \boldsymbol{0} \tag{4}$$

Scattered Fields at Observation Point $\boldsymbol{r}$

(1), (2), (3-a), (4) →

$$\boldsymbol{E}_S(\boldsymbol{r}) = -j\frac{\eta}{k}\left[k^2\iint_S \boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')dS' + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\iint_S \boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')dS'\right)\right] \tag{5-a}$$

$$\boldsymbol{H}_S(\boldsymbol{r}) = \boldsymbol{\nabla}\times\iint_S \boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')dS' \tag{5-b}$$

(5) →

$$\boldsymbol{E}_S(\boldsymbol{r}) = -j\frac{\eta}{k}\left[k^2\iint_S \boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')dS' + \boldsymbol{\nabla}\left(\iint_S \boldsymbol{\nabla}'\cdot\boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')dS'\right)\right] \tag{6-a}$$

$$\boldsymbol{H}_S(\boldsymbol{r}) = \iint_S \boldsymbol{\nabla}'\times[\boldsymbol{J}_s(\boldsymbol{r}_b')G(\boldsymbol{r},\boldsymbol{r}_b')]dS' \tag{6-b}$$

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$$\nabla' \times \boldsymbol{J}_s(\boldsymbol{r}'_b) = \boldsymbol{0} \tag{7}$$

(VI-1), (6-b), (7) → $$\boldsymbol{H}_S(\boldsymbol{r}) = -\iint_S \boldsymbol{J}_s(\boldsymbol{r}'_b) \times \nabla' G(\boldsymbol{r}, \boldsymbol{r}'_b) dS' \tag{8}$$

(3-c), (6-a) → $$\boldsymbol{n} \times \left\{ -j\frac{\eta}{k} \left[k^2 \iint_S \boldsymbol{J}_s(\boldsymbol{r}'_b) G(\boldsymbol{r}_b, \boldsymbol{r}'_b) dS' + \nabla \left(\iint_S \nabla' \cdot \boldsymbol{J}_s(\boldsymbol{r}'_b) G(\boldsymbol{r}_b, \boldsymbol{r}'_b) dS' \right) \right] \right\} = -\boldsymbol{n} \times \boldsymbol{E}_{inc}(\boldsymbol{r} = \boldsymbol{r}_b) \tag{9}$$

(3-d), (8) → $$\boldsymbol{n} \times \iint_S \boldsymbol{J}_s(\boldsymbol{r}'_b) \times \nabla' G(\boldsymbol{r}_b, \boldsymbol{r}'_b) dS' + \boldsymbol{J}_s(\boldsymbol{r}_b) = \boldsymbol{n} \times \boldsymbol{H}_{inc}(\boldsymbol{r} = \boldsymbol{r}_b) \tag{10}$$

Toy Problems

2D EFIE for TM^z Polarized Wave, Incident on an Infinite PEC Cylinder

Assumptions and Corollaries

Scattered wave is TM^z Polarized. (11-a)

$$\boldsymbol{J}_s(\boldsymbol{\rho}_b) = J_s(\boldsymbol{\rho}_b)\boldsymbol{a}_z \tag{11-b}$$

(11-b) → $$\nabla \cdot \boldsymbol{J}_s(\boldsymbol{\rho}_b) = 0 \tag{11-c}$$

Green's Function in Polar Coordinates

$$\begin{aligned} \boldsymbol{\rho} - \boldsymbol{\rho}' &\equiv \rho \boldsymbol{a}_\rho - \rho' \boldsymbol{a}_{\rho'} \\ &= (\rho\cos\phi - \rho'\cos\phi')\boldsymbol{a}_x + (\rho\sin\phi - \rho'\sin\phi')\boldsymbol{a}_y \end{aligned} \tag{12}$$

(12) → $$\begin{aligned} |\boldsymbol{r} - \boldsymbol{r}'| &\equiv \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ &= \sqrt{|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2 + (z-z')^2} \end{aligned} \tag{13}$$

(2-c), (13) → $$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{e^{-jk\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + (z-z')^2}}}{4\pi\sqrt{|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2 + (z-z')^2}} \tag{14}$$

$$\int_{-\infty}^{+\infty} G(\boldsymbol{\rho}, \boldsymbol{\rho}')\, dz' = -j\frac{1}{4} H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \tag{15}$$

On curve C, the normal vector $\boldsymbol{n}$ and the tangential vector $\boldsymbol{a}_c$ satisfy $\boldsymbol{n} \times \boldsymbol{a}_c = \boldsymbol{a}_z$. (16)

(9), (11-c) → $$\boldsymbol{n} \times \left\{ j\eta k \iint_S \boldsymbol{J}_s(\boldsymbol{\rho}'_b) G(\boldsymbol{\rho}_b, \boldsymbol{\rho}'_b) dS' \right\} = \boldsymbol{n} \times \boldsymbol{E}_{inc}(\boldsymbol{\rho} = \boldsymbol{\rho}_b) \tag{17}$$

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$$\iint_S \boldsymbol{J}_s(\boldsymbol{\rho}'_b)G(\boldsymbol{\rho}_b,\boldsymbol{\rho}'_b)dS' = \int_C \boldsymbol{J}_s(\boldsymbol{\rho}'_b)\left[\int_{-\infty}^{+\infty} G(\boldsymbol{\rho},\boldsymbol{\rho}')\,dz'\right]dC'$$

(11-b), (15) →

$$= -j\frac{1}{4}\int_C \boldsymbol{J}_s(\boldsymbol{\rho}'_b)H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\,dC' \quad (18)$$

$$= \left(-j\frac{1}{4}\int_C J_s(\boldsymbol{\rho}'_b)H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\,dC'\right)a_z$$

(16), (17), (18) →

$$\frac{\eta k}{4}\int_C J_s(\boldsymbol{\rho}'_b)H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\,dC' = E_{inc}(\boldsymbol{\rho}=\boldsymbol{\rho}_b)$$

2D MFIE for TE^z Polarized Wave, Incident on an Infinite PEC Cylinder

Assumptions and Corollaries

Scattered wave is TE^z Polarized. (19-a)

$\boldsymbol{J}_s(\boldsymbol{\rho}_b) = J_s(\boldsymbol{\rho}_b)\boldsymbol{a}_c$ (19-b)

Green's Function in Polar Coordinates

$$\boldsymbol{\rho}-\boldsymbol{\rho}' \equiv \rho\boldsymbol{a}_\rho - \rho'\boldsymbol{a}_{\rho'} = (\rho\cos\phi-\rho'\cos\phi')\boldsymbol{a}_x + (\rho\sin\phi-\rho'\sin\phi')\boldsymbol{a}_y \quad (20)$$

(20) →

$$|\boldsymbol{r}-\boldsymbol{r}'| \equiv \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2} \quad (21)$$

(2-c), (21) →

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{e^{-jk\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}}}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}} \quad (22)$$

$$\int_{-\infty}^{+\infty} G(\boldsymbol{\rho},\boldsymbol{\rho}')\,dz' = -j\frac{1}{4}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) \quad (23)$$

$$\boldsymbol{A}\times\boldsymbol{B}\times\boldsymbol{C} = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \quad \text{BCR}$$

On curve C, the normal vector $\boldsymbol{n}$ and the tangential vector $\boldsymbol{a}_c$ satisfy $\boldsymbol{n}\times\boldsymbol{a}_c = \boldsymbol{a}_z$. (24)

(10), (23) →

$$\iint_S \boldsymbol{J}_s(\boldsymbol{\rho}'_b)\times\boldsymbol{\nabla}'G(\boldsymbol{\rho}_b,\boldsymbol{\rho}'_b)dS' = \int_C \boldsymbol{J}_s(\boldsymbol{\rho}'_b)\times\boldsymbol{\nabla}'\left[\int_{-\infty}^{+\infty} G(\boldsymbol{\rho}_b,\boldsymbol{\rho}'_b)\,dz'\right]dC'$$

$$= -j\frac{1}{4}\int_C \boldsymbol{J}_s(\boldsymbol{\rho}'_b)\times\boldsymbol{\nabla}'H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\,dC' \quad (25)$$

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(10), (25) → $$-j\frac{1}{4}\left(\boldsymbol{n}\times\int_{C}\boldsymbol{J}_s(\boldsymbol{\rho}_b')\times\boldsymbol{\nabla}'H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\,dC'\right)+\boldsymbol{J}_s(\boldsymbol{\rho}_b)=\boldsymbol{n}\times\boldsymbol{H}_{inc}(\boldsymbol{\rho}=\boldsymbol{\rho}_b) \quad (26)$$

(26), BCR → $$J_s(\boldsymbol{\rho}_b)-j\frac{1}{4}\int_{C}J_s(\boldsymbol{\rho}_b')\left(\boldsymbol{\nabla}'H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\cdot\boldsymbol{n}'\right)\,dC'=-H_{inc}(\boldsymbol{\rho}=\boldsymbol{\rho}_b)$$

Table 37. Derivation of the 2D and 1D forms of the General Solution to the Poisson's Equation

General Solution to the Poisson's Equation

$$V(\boldsymbol{r})=\frac{1}{\varepsilon}\iiint_{V'}\rho(\boldsymbol{r}')\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}dv' \quad (1)$$

Toy Problems

2D Conducting Plate

$$\boldsymbol{r}=x\boldsymbol{a}_x+y\boldsymbol{a}_y \quad (2\text{-a})$$
$$\boldsymbol{r}'=x'\boldsymbol{a}_x+y'\boldsymbol{a}_y \quad (2\text{-b})$$

$$\rho(\boldsymbol{r}')=\rho(x',y')\delta(z') \quad (3)$$

(1), (2), (3) →
$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\iiint_{V'}\frac{\rho(x',y')\delta(z')}{\sqrt{(x-x')^2+(y-y')^2}}dz'dx'dy'$$
$$= \frac{1}{4\pi\varepsilon}\iint\rho(x',y')\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}dx'dy'$$

1D Conducting Rod

$$\boldsymbol{r}=x\boldsymbol{a}_x+h\boldsymbol{a}_z \quad (4\text{-a})$$
$$\boldsymbol{r}'=x'\boldsymbol{a}_x \quad (4\text{-b})$$

$$\rho(\boldsymbol{r}')=\rho(x')\delta(y')\delta(z') \quad (5)$$

(1), (4), (5) →
$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\iiint_{V'}\frac{\rho(x')\delta(y')\delta(z')}{\sqrt{(x-x')^2+h^2}}dz'dx'dy'$$
$$= \frac{1}{4\pi\varepsilon}\int\rho(x')\frac{1}{\sqrt{(x-x')^2+h^2}}dx'$$

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Table 38. Derivation of the System of Equations for 2D and 1D MoM Problems

Weighted-Residual Method

To find the solution of $\mathcal{L}(\psi(\boldsymbol{r}'),\boldsymbol{r}) = f(\boldsymbol{r})$ over the domain Ω, in which $\mathcal{L}(\cdot,\boldsymbol{r})$ is a ***linear integral operator***, the domain of interest is divided into non-overlapping segments Ω_e^i over which the unknown solution $\psi(\boldsymbol{r}')$ is approximated by a ***basis function*** $\phi_i(\boldsymbol{r}')$ that is scaled by a scaling factor ψ_i (ϕ_i is non-zero over Ω_e^i and is zero everywhere else over Ω). This scheme transforms the original problem to that of constructing a system of linear equations for the scaling factors in the entire domain Ω.

A system of linear equations can be set up by setting $\int_{\Omega_e^i} w_i(\boldsymbol{r}) R\, d\Omega = 0$ over each Ω_e^i in which $R \equiv \mathcal{L}(\psi(\boldsymbol{r}'),\boldsymbol{r}) - f(\boldsymbol{r})$ is called the ***residual*** and $w_i(\boldsymbol{r})$ are called the ***weighting*** (testing) functions. In the ***Point Matching*** (Collocation) variant of this method, $w_i(\boldsymbol{r}) = \delta(\boldsymbol{r} - \boldsymbol{r}_i)$ in which $\boldsymbol{r}_i$ is the location of the i^{th} observation point.

Discretization of the Solution

$$\psi(\boldsymbol{r}) \approx \sum_{j=1}^{N} \psi_j \phi_j(\boldsymbol{r}) \tag{1}$$

The Linear Equation Resulting from the Weighted-Residual Integration

$$\begin{aligned} \iint_{S'} w_i(\boldsymbol{r}) R\, dS' &= \iint_{S'} \delta(\boldsymbol{r} - \boldsymbol{r}_i)[\mathcal{L}(\psi(\boldsymbol{r}'),\boldsymbol{r}) - f(\boldsymbol{r})]\, dS' \\ &= \mathcal{L}(\psi(\boldsymbol{r}'),\boldsymbol{r} = \boldsymbol{r}_i) - f(\boldsymbol{r} = \boldsymbol{r}_i) \\ &= 0 \end{aligned} \tag{2}$$

$$(2) \quad \rightarrow \quad \mathcal{L}(\psi(\boldsymbol{r}'),\boldsymbol{r} = \boldsymbol{r}_i) = f(\boldsymbol{r} = \boldsymbol{r}_i) \tag{3}$$

$$(1), (3) \quad \rightarrow \quad \mathcal{L}\left(\sum_{j=1}^{N} \psi_j \phi_j(\boldsymbol{r}')\,, \boldsymbol{r} = \boldsymbol{r}_i\right) = f(\boldsymbol{r} = \boldsymbol{r}_i) \tag{4}$$

$$(4) \quad \rightarrow \quad \sum_{j=1}^{N} \psi_j \mathcal{L}\big(\phi_j(\boldsymbol{r}'),\boldsymbol{r} = \boldsymbol{r}_i\big) = f(\boldsymbol{r} = \boldsymbol{r}_i) \tag{5}$$

(5) → To construct the linear equation corresponding to Ω_e^i, the coefficients of the unknown ψ_j can be computed by evaluating $\mathcal{L}\big(\phi_j(\boldsymbol{r}'),\boldsymbol{r} = \boldsymbol{r}_i\big)$ and the right-hand-side of the equation can be computed using $f(\boldsymbol{r} = \boldsymbol{r}_i)$. (6)

Toy Problems using Piecewise-Constant Basis Function

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$$\frac{\eta k}{4}\int_C J_s(\boldsymbol{\rho}'_b)H_0^{(2)}(k|\boldsymbol{\rho}_b-\boldsymbol{\rho}'_b|)\,dC' = E_{inc}(\boldsymbol{\rho}=\boldsymbol{\rho}_b) \tag{7}$$

$$\boldsymbol{E}_{inc}(\boldsymbol{r}) = \left(E_0 e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{r}}\right)\boldsymbol{a}_z \tag{8}$$

(6), (7), (8) →

$$\mathcal{L}\left(\phi_j(\boldsymbol{r}'),\boldsymbol{r}=\boldsymbol{r}_i\right) = \frac{\eta k}{4}\int_{\Omega_e^j} H_0^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}'_b\right|\right)dC' \tag{9-a}$$

$$f(\boldsymbol{r}=\boldsymbol{r}_i) = E_0 e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{\rho}_{b,i}} \tag{9-b}$$

Caring for Singularities

In numerical evaluation of (9-a), when Ω_e^j contains $\boldsymbol{\rho}_{b,i}$, the Hankel function goes to infinity as its argument approaches zero causing erroneous results. Therefore, the integral must be evaluated analytically. (10)

The Small Argument Approximation for the Hankel Function

$$H_0^{(2)}(kR) \approx 1 - j\frac{2}{\pi}\ln\left(\frac{\gamma k}{2}R\right) \qquad \gamma = 1.781 \tag{11}$$

$$\begin{aligned}\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}'_b &\equiv \rho_b\left(\boldsymbol{a}_{\rho,i}-\boldsymbol{a}_{\rho'}\right)\\ &= \rho_b\left[(\cos\phi_i-\cos\phi')\boldsymbol{a}_x+(\sin\phi_i-\sin\phi')\boldsymbol{a}_y\right]\end{aligned} \tag{12}$$

The observation point is chosen to be at the middle of the interval, i.e., $\phi_i \equiv \phi_a + \frac{1}{2}\Delta\phi$. (13)

(9-a), (10), (11) →

$$\begin{aligned}I_0 &\equiv \int_{\Omega_e^j} H_0^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}'_b\right|\right)dC'\\ &\approx \int_{\phi_a}^{\phi_a+\Delta\phi} 1-j\frac{2}{\pi}\ln\left(\frac{\gamma k}{2}\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}'_b\right|\right)d\phi'\\ &= \Delta\phi - j\frac{2}{\pi}I_1\end{aligned} \tag{14}$$

(12), (14) →

$$\begin{aligned}I_1 &\equiv \int_{\phi_a}^{\phi_a+\Delta\phi} \ln\left(\frac{\gamma k}{2}\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}'_b\right|\right)d\phi'\\ &= \int_{\phi_a}^{\phi_a+\Delta\phi} \ln\left(\frac{\gamma k\rho_b}{\sqrt{2}}\sqrt{1-\cos(\phi'-\phi_i)}\right)d\phi'\end{aligned} \tag{15}$$

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$$= \Delta\phi \ln(\gamma k \rho_b) + I_2$$

$$I_2 \equiv \int_{\phi_a}^{\phi_a+\Delta\phi} \ln\left(\left|\sin\left(\frac{1}{2}\phi' - \frac{1}{2}\phi_i\right)\right|\right) d\phi'$$

(13), (15) → $$= 2\int_{-\frac{\Delta\phi}{4}}^{\frac{\Delta\phi}{4}} \ln(|\sin u|)\ du \qquad (16)$$

$$= 4\int_{0}^{\frac{\Delta\phi}{4}} \ln(\sin u)\ du$$

$$\int \ln(\sin u)\ du \approx u \ln u - u - \frac{u^3}{18} - \frac{u^5}{900} \text{ for } 0 < u < \pi \qquad (17)$$

(14), (15), (16), (17) → $$I_0 = \Delta\phi\left[1 + j\frac{2}{\pi}\left(1 + \frac{1}{288}(\Delta\phi)^2 - \ln\left(\frac{\gamma k \rho_b}{4}\Delta\phi\right)\right)\right]$$

2D MFIE for TE^z Polarized Wave, Incident on an Infinite PEC Cylinder

$$J_s(\boldsymbol{\rho}_b) - j\frac{1}{4}\int_C J_s(\boldsymbol{\rho}_b')\left(\boldsymbol{\nabla}' H_0^{(2)}(k|\boldsymbol{\rho}_b - \boldsymbol{\rho}_b'|)\cdot \boldsymbol{n}'\right) dC' = -H_{inc}(\boldsymbol{\rho} = \boldsymbol{\rho}_b) \qquad (18)$$

$$\boldsymbol{E}_{inc}(\boldsymbol{r}) = \left(E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y\right)e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{r}} \qquad (19\text{-a})$$

$$\boldsymbol{H}_{inc}(\boldsymbol{r}) = \frac{1}{\eta}\boldsymbol{a}_{inc} \times \boldsymbol{E}_{inc}(\boldsymbol{r}) \qquad (19\text{-b})$$

$$\frac{d}{du}H_0^{(2)}(\alpha u) = -\alpha H_1^{(2)}(\alpha u) \qquad \text{BI}$$

$$J_s(\boldsymbol{\rho}_b) + j\frac{k}{4}\int_C J_s(\boldsymbol{\rho}_b')H_1^{(2)}(k|\boldsymbol{\rho}_b - \boldsymbol{\rho}_b'|)\cos\xi'\, dC' = -H_{inc}(\boldsymbol{\rho} = \boldsymbol{\rho}_b) \qquad (20\text{-a})$$

(18), BI → $$\cos\xi' \equiv \frac{\left(\boldsymbol{a}_{\rho,i} - \boldsymbol{a}_{\rho'}\right)\cdot \boldsymbol{a}_{\rho'}}{\left|\boldsymbol{a}_{\rho,i} - \boldsymbol{a}_{\rho'}\right|} = -\frac{1}{\sqrt{2}}\sqrt{1 - \cos(\phi' - \phi_i)} \qquad (20\text{-b})$$

(6), (19), (20) → $\mathcal{L}\left(\phi_j(\boldsymbol{r}'), \boldsymbol{r} = \boldsymbol{r}_i\right) = j\frac{k}{4}\int_{\Omega_e^j} H_1^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i} - \boldsymbol{\rho}_b'\right|\right)\cos\xi'\, dC'$ for $j \neq i$ and $\mathcal{L}\left(\phi_j(\boldsymbol{r}'), \boldsymbol{r} = \boldsymbol{r}_i\right) = 1 + j\frac{k}{4}\int_{\Omega_e^j} H_1^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i} - \boldsymbol{\rho}_b'\right|\right)\cos\xi'\, dC'$ for $j = i$. (21-a)

$$f(\boldsymbol{r} = \boldsymbol{r}_i) = -\frac{1}{\eta}\left(E_{0y}\cos\phi_{inc} - E_{0x}\sin\phi_{inc}\right)e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{\rho}_{b,i}} \qquad (21\text{-b})$$

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In numerical evaluation of (21-a), when Ω_e^j contains $\boldsymbol{\rho}_{b,i}$, the Hankel function goes to infinity as its argument approaches zero causing erroneous results. Furthermore, the expression $I_0 \equiv j\frac{k}{4}\int_{\Omega_e^j} H_1^{(2)}\left(k\left|\boldsymbol{\rho}_{b,i}-\boldsymbol{\rho}_b'\right|\right)\cos\xi'\,dC'$, which represents the scattered magnetic field, must be evaluated at observation points that are outside the imaginary surface S and are infinitesimally close to it (but are not on it!); In other words, the above expression must be evaluated in the form of a limit as $\boldsymbol{\rho}_{b,i}$ approaches S from outside.

The Physical Equivalence Theorem requires that the scattering current density $\boldsymbol{J}_s$ radiate $-\boldsymbol{H}_{inc}$ inside the imaginary surface S. Noting that $S=\bigcup_{j=1}^{N}\Omega_e^j$, at any segment Ω_e^j, this is realized by the superposition of two scattered field components propagating inside of S. The first, denoted by $\boldsymbol{H}_S^{\Omega_e^j,-}$, represents the scattered magnetic field due to $\boldsymbol{J}_s^{\Omega_e^j}$ (i.e., $\boldsymbol{J}_s$ at Ω_e^j). The second, denoted by $\boldsymbol{H}_S^{\left(S-\Omega_e^j\right),-}$, represents the total scattered magnetic field due to $\boldsymbol{J}_s$ from all other segments of S. Therefore, the requirement that $-\boldsymbol{H}_{inc}$ propagate inside S is satisfied by the simple mechanism of $\boldsymbol{J}_s^{\Omega_e^j}$ radiating scattered magnetic fields at Ω_e^j. (22)

The scattering current density $\boldsymbol{J}_s^{\Omega_e^j}$ radiates scattered magnetic fields both inside and outside of S. The scattered field radiated outside, denoted by $\boldsymbol{H}_S^{\Omega_e^j,+}$, is related to the scattered field radiated inside by $\boldsymbol{H}_S^{\Omega_e^j,+}=-\boldsymbol{H}_S^{\Omega_e^j,-}$. Therefore, from $\boldsymbol{n}\times\left(\boldsymbol{H}_S^{\Omega_e^j,+}-\boldsymbol{H}_S^{\Omega_e^j,-}\right)=\boldsymbol{J}_s^{\Omega_e^j}$ it follows that $\boldsymbol{n}\times\boldsymbol{H}_S^{\Omega_e^j,+}=\frac{1}{2}\boldsymbol{J}_s^{\Omega_e^j}$.

Going back to the problem of evaluating I_0, from the above discussion, the integral (which represents the scattered magnetic field) can be replaced with $-\frac{1}{2}J_s^{\Omega_e^j}$.

(21-a), (22) $\rightarrow$ $\mathcal{L}\left(\phi_j(\boldsymbol{r}'),\boldsymbol{r}=\boldsymbol{r}_i\right)=\frac{1}{2}$ for $j=i$.

2D Electrostatic Charge Distribution

$$\frac{1}{4\pi\varepsilon}\iint\rho(x',y')\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}dx'dy'=V(\boldsymbol{r}) \tag{23}$$

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$$V(\boldsymbol{r}) = V_0 \tag{24}$$

(6), (23), (24) →

$$\mathcal{L}\big(\phi_j(\boldsymbol{r}'), \boldsymbol{r} = \boldsymbol{r}_i\big) = \frac{1}{4\pi\varepsilon} \iint\limits_{\Omega_e^j} \frac{1}{\sqrt{(x_i - x')^2 + (y_i - y')^2}} dx'dy' \tag{25-a}$$

$$f(\boldsymbol{r} = \boldsymbol{r}_i) = V_0 \tag{25-b}$$

Caring for Singularities

In numerical evaluation of (30-a), when Ω_e^j contains $\boldsymbol{r}_i$, the integrand goes to infinity as its denominator approaches zero causing erroneous results. Therefore, the integral must be evaluated analytically. (26)

The observation point is chosen to be at the middle of the x and y intervals, i.e., $x_i \equiv x_a + \frac{1}{2}\Delta$ and $y_i \equiv y_a + \frac{1}{2}\Delta$ with the assumption that $\Delta y = \Delta x = \Delta$. (27)

(25-a), (26), (27) →

$$\begin{aligned} I_0 &\equiv \iint\limits_{\Omega_e^j} \frac{1}{\sqrt{(x_i - x')^2 + (y_i - y')^2}} dx'dy' \\ &= \int\limits_{y_a}^{y_a+\Delta} \left[\int\limits_{x_a}^{x_a+\Delta} \frac{1}{\sqrt{(x_i - x')^2 + (y_i - y')^2}}\, dx' \right] dy' \\ &= \int\limits_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \left[\int\limits_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \frac{1}{\sqrt{u^2 + v^2}}\, du \right] dv \end{aligned} \tag{28}$$

$$\int \frac{1}{\sqrt{u^2 + a^2}}\, du = \frac{1}{2} \ln\left(\frac{\sqrt{u^2 + a^2} + u}{\sqrt{u^2 + a^2} - u} \right) \tag{29}$$

(28), (29) →

$$I_0 = \int\limits_{-\frac{1}{2}\Delta}^{\frac{1}{2}\Delta} \ln\left(\frac{\sqrt{v^2 + \left(\frac{\Delta}{2}\right)^2} + \frac{\Delta}{2}}{\sqrt{v^2 + \left(\frac{\Delta}{2}\right)^2} - \frac{\Delta}{2}} \right) dv \tag{30}$$

$$\begin{aligned} \int \ln\left(\frac{\sqrt{v^2 + a^2} + a}{\sqrt{v^2 + a^2} - a} \right) dv &= v \ln\left(\frac{\sqrt{v^2 + a^2} + a}{\sqrt{v^2 + a^2} - a} \right) \\ &+ 2a \ln\left(\sqrt{v^2 + a^2} + v \right) \end{aligned} \tag{31}$$

(30), (31) →

$$I_0 = 2\Delta \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

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1D Electrostatic Charge Distribution

$$\frac{1}{4\pi\varepsilon}\int \rho(x')\frac{1}{\sqrt{(x-x')^2+h^2}}dx' = V(\boldsymbol{r}) \tag{32}$$

$$V(\boldsymbol{r}) = V_0 \tag{33}$$

(6), (32), (33) →

$$\mathcal{L}\big(\phi_j(\boldsymbol{r}'), \boldsymbol{r}=\boldsymbol{r}_i\big) = \frac{1}{4\pi\varepsilon}\int_{\Omega_e^j}\frac{1}{\sqrt{(x_i-x')^2+h^2}}dx'$$

$$f(\boldsymbol{r}=\boldsymbol{r}_i) = V_0$$

Table 39. Derivation of the Analytical Solution for the Scattered Current Density

$$\begin{aligned} e^{-jkx} &= e^{-jk\rho\cos\phi} \\ &= \sum_{n=-\infty}^{+\infty}(j)^{-n}J_n(k\rho)e^{jn\phi} \end{aligned} \tag{I-1}$$

$$\frac{2}{\pi\alpha\rho} = J_n(\alpha\rho)Y_n'(\alpha\rho) - J_n'(\alpha\rho)Y_n(\alpha\rho) \tag{I-2}$$

The TM^z Mode

$$\boldsymbol{H} = -\frac{1}{j\omega\mu}\boldsymbol{\nabla}\times\boldsymbol{E} \tag{1}$$

The Incident Electric and Magnetic Fields

$$\boldsymbol{E}_{inc} = \left(E_0 e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{r}}\right)\boldsymbol{a}_z \tag{2-a}$$

$$\boldsymbol{a}_{inc} = \boldsymbol{a}_x \tag{2-b}$$

$$\begin{aligned} E_{inc,z} &= E_0 e^{-jkx} \\ &= E_0\sum_{n=-\infty}^{+\infty}(j)^{-n}J_n(k\rho)e^{jn\phi} \end{aligned} \tag{3-a}$$

I-1, (1), (2) →

$$\begin{aligned} H_{inc,\rho} &= -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_{inc,z}}{\partial\phi} \\ &= -\frac{E_0}{j\omega\mu}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty}n(j)^{-n+1}J_n(k\rho)e^{jn\phi} \end{aligned} \tag{3-b}$$

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$$H_{inc,\phi} = \frac{1}{j\omega\mu}\frac{\partial E_{inc,z}}{\partial \rho} = \frac{kE_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty}(j)^{-n}J_n'(k\rho)e^{jn\phi} \quad (3\text{-c})$$

The Scattered Electric Field

$$E_{s,z} = E_0\sum_{n=-\infty}^{+\infty}c_n H_n^{(2)}(k\rho) \quad (4)$$

The Total Electric Field

$$E_{t,z} = E_{inc,z} + E_{s,z} = E_0\sum_{n=-\infty}^{+\infty}\left((j)^{-n}J_n(k\rho)e^{jn\phi} + c_n H_n^{(2)}(k\rho)\right) \quad (5)$$

The Boundary Condition for the Electric Field

$$E_{t,z}(\rho=\rho_b,\phi,z) = 0 \quad (6)$$

(5), (6) → $$c_n = -(j)^{-n}\frac{J_n(k\rho_b)}{H_n^{(2)}(k\rho_b)}e^{jn\phi} \quad (7)$$

(4), (7) → $$E_{s,z} = -E_0\sum_{n=-\infty}^{+\infty}(j)^{-n}\frac{J_n(k\rho_b)}{H_n^{(2)}(k\rho_b)}H_n^{(2)}(k\rho)e^{jn\phi} \quad (8\text{-a})$$

(1), (8-a) → $$H_{s,\rho} = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_{s,z}}{\partial\phi} = \frac{E_0}{j\omega\mu}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty}n(j)^{-n+1}\frac{J_n(k\rho_b)}{H_n^{(2)}(k\rho_b)}H_n^{(2)}(k\rho)e^{jn\phi} \quad (8\text{-b})$$

(1), (8-a) → $$H_{s,\phi} = \frac{1}{j\omega\mu}\frac{\partial E_{s,z}}{\partial\rho} = -\frac{kE_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty}(j)^{-n}\frac{J_n(k\rho_b)}{H_n^{(2)}(k\rho_b)}H_n^{(2)'}(k\rho)e^{jn\phi} \quad (8\text{-c})$$

The Current Density

$$\boldsymbol{J} = \boldsymbol{n}\times\boldsymbol{H}_t(\rho=\rho_b) = H_{t,\phi}(\rho=\rho_b)\boldsymbol{a}_z \quad (9)$$

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(3-c), (8-c) → $$H_{t,\phi} = \frac{kE_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty}(j)^{-n}\left(J_n'(k\rho) - \frac{J_n(k\rho_b)}{H_n^{(2)}(k\rho_b)}H_n^{(2)'}(k\rho)\right)e^{jn\phi} \quad (10)$$

I-2, (9), (10) → $$J_z(\phi) = \frac{2E_0}{\pi k\eta\rho_b}\sum_{n=-\infty}^{+\infty}(j)^{-n}\frac{1}{H_n^{(2)}(k\rho_b)}e^{jn\phi}$$

The TE^z Mode

$$\boldsymbol{E} = \frac{1}{j\omega\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{H} \quad (11)$$

The Incident Electric and Magnetic Fields

$$\boldsymbol{H}_{inc} = \left(H_0 e^{-jk\boldsymbol{a}_{inc}\cdot\boldsymbol{r}}\right)\boldsymbol{a}_z \quad (12\text{-a})$$

$$\boldsymbol{a}_{inc} = \boldsymbol{a}_x \quad (12\text{-b})$$

$$\begin{aligned} H_{inc,z} &= H_0 e^{-jkx} \\ &= H_0\sum_{n=-\infty}^{+\infty}(j)^{-n}J_n(k\rho)e^{jn\phi} \end{aligned} \quad (13\text{-a})$$

I-1, (11), (12-a) →

$$\begin{aligned} E_{inc,\rho} &= \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_{inc,z}}{\partial\phi} \\ &= \frac{H_0}{j\omega\varepsilon}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty}n(j)^{-n+1}J_n(k\rho)e^{jn\phi} \end{aligned} \quad (13\text{-b})$$

$$\begin{aligned} E_{inc,\phi} &= -\frac{1}{j\omega\varepsilon}\frac{\partial H_{inc,z}}{\partial\rho} \\ &= -\frac{kH_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty}(j)^{-n}J_n'(k\rho)e^{jn\phi} \end{aligned} \quad (13\text{-c})$$

The Scattered Magnetic Field

$$H_{s,z} = H_0\sum_{n=-\infty}^{+\infty}d_n H_n^{(2)}(k\rho) \quad (14)$$

The Scattered Electric Field

$$\begin{aligned} E_{s,\rho} &= \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_{s,z}}{\partial\phi} \\ &= \frac{H_0}{j\omega\varepsilon}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty}\frac{\partial d_n}{\partial\phi}H_n^{(2)}(k\rho) \end{aligned} \quad (15\text{-a})$$

(11), (14)

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$$E_{s,\phi} = -\frac{1}{j\omega\varepsilon}\frac{\partial H_{inc,z}}{\partial \rho} = -\frac{kH_0}{j\omega\varepsilon}\sum_{n=-\infty}^{+\infty} d_n H_n^{(2)'}(k\rho) \quad (15\text{-b})$$

The Total Tangential Electric Field

(13-c), (15-b) →

$$E_{t,\phi} = E_{inc,\phi} + E_{s,\phi} = -\frac{kH_0}{j\omega\varepsilon}\sum_{n=-\infty}^{+\infty}\left((j)^{-n}J_n'(k\rho)e^{jn\phi} + d_n H_n^{(2)'}(k\rho)\right) \quad (16)$$

The Boundary Condition for the Electric Field

$$E_{t,\phi}(\rho = \rho_b, \phi, z) = 0 \quad (17)$$

(16), (17) →

$$d_n = -(j)^{-n}\frac{J_n'(k\rho_b)}{H_n^{(2)'}(k\rho_b)}e^{jn\phi} \quad (18)$$

The Current Density

$$\boldsymbol{J} = \boldsymbol{n}\times\boldsymbol{H}_t(\rho=\rho_b) = -H_{t,z}(\rho=\rho_b)\boldsymbol{a}_\phi \quad (19)$$

(13-a), (14) →

$$H_{t,z} = H_0\sum_{n=-\infty}^{+\infty}(j)^{-n}\left(J_n(k\rho) - \frac{J_n'(k\rho_b)}{H_n^{(2)'}(k\rho_b)}H_n^{(2)}(k\rho)\right)e^{jn\phi} \quad (20)$$

I-2, (19), (20) →

$$J_z(\phi) = j\frac{2H_0}{\pi k\rho_b}\sum_{n=-\infty}^{+\infty}(j)^{-n}\frac{1}{H_n^{(2)'}(k\rho_b)}e^{jn\phi}$$

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14- Radiation of a Line Source

Table 40. Far-Field Radiation of an Infinite Constant Electric Current

The TM^z Mode

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\varepsilon}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$$

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{A}$$

$$\boldsymbol{A} = A_z\boldsymbol{a}_z$$

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

The Field Configuration

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z} \qquad H_\rho = -\frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}$$
$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z} \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} \tag{1}$$
$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon\right)A_z \qquad H_z = 0$$

The Current Density Distribution

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\left(\frac{1}{\rho}\delta(\phi-\phi_0)\right)\boldsymbol{a}_z \tag{2}$$

The Vector Potential

Some Useful Math

The position vector $\boldsymbol{r}$ can be expressed in cylindrical coordinates as $\boldsymbol{r} = \boldsymbol{\rho} + z\boldsymbol{a}_z$ in which $\boldsymbol{\rho} = \rho\boldsymbol{a}_\rho$. Therefore, $|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + (z-z')^2}$. (3-a)

$$\int_{-\infty}^{+\infty}\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}dz'$$

(3-a) → (3-b)

$$= \int_{-\infty}^{+\infty}\frac{e^{-jk\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}}}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}}dz'$$
$$= -j\frac{1}{4}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)$$

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$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{r}) &= \mu \iiint_{V'} \boldsymbol{J}(\boldsymbol{r}') \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} dv' \\
&= \mu \iiint_{V'} \left[I_0 \delta(\rho'-\rho_0) \left(\frac{1}{\rho} \delta(\phi'-\phi_0) \right) \boldsymbol{a}_z \right] \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} dv' \\
(2),(3) \quad \rightarrow \quad &= \mu I_0 \left(\iiint_{V'} \delta(\rho'-\rho_0) \left(\frac{1}{\rho'} \delta(\phi'-\phi_0) \right) \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} dz'(\rho' d\phi') d\rho' \right) \boldsymbol{a}_z \qquad (4) \\
&= \mu I_0 \left(\left[\int \delta(\rho'-\rho_0) \left\{ \int \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} dz' \right\} d\rho' \right] \left[\int \delta(\phi'-\phi_0) d\phi' \right] \right) \boldsymbol{a}_z \\
&= -j \frac{\mu I_0}{4} H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|) \boldsymbol{a}_z
\end{aligned}
$$

Since $A_z = -j\frac{\mu I_0}{4} H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|)$ is independent of the z coordinate, all variations w.r.t the z coordinate are zero, i.e., $\frac{\partial}{\partial z} = 0$. Therefore, $E_\rho = 0$, $E_\phi = 0$ and $E_z = -j\omega A_z$.

Furthermore, To obtain H_ρ and H_ϕ, in the first step, the quantity R is defined as $R \equiv |\boldsymbol{\rho}-\boldsymbol{\rho}_0| = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\Delta\phi}$ in which $\Delta\phi \equiv \phi-\phi_0$; In the second step, $\frac{\partial A_z}{\partial \rho} = \frac{\partial A_z}{\partial R}\frac{\partial R}{\partial \rho}$ and $\frac{\partial A_z}{\partial \phi} = \frac{\partial A_z}{\partial R}\frac{\partial R}{\partial \phi}$. It can be shown that, $\frac{\partial R}{\partial \rho} = \frac{\rho-\rho_0\cos\Delta\phi}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}$ and that $\frac{\partial R}{\partial \phi} = \frac{\rho\rho_0\sin\Delta\phi}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}$.

(1), (4) →

$$E_z = -\frac{k\eta}{4} I_0 H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|) \qquad (5\text{-a})$$

$$H_\rho = j\frac{k}{4} I_0 \frac{H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|)}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}(\rho_0 \sin\Delta\phi) \qquad (5\text{-b})$$

$$H_\phi = -j\frac{k}{4} I_0 \frac{H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|)}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}(\rho - \rho_0\cos\Delta\phi) \qquad (5\text{-c})$$

The Radiated Fields in the Far-Field Region

For large values of its argument, the Hankel function can be approximated using an exponential, i.e., $H_n^{(2)}(x) \approx \left(\sqrt{\frac{2}{\pi}} e^{j(2n+1)\frac{\pi}{4}}\right)\frac{1}{\sqrt{x}} e^{-jx}$. (6-a)

Using the Tylor expansion, $R = \rho\left(1+\left(\frac{\rho_0}{\rho}\right)^2 - 2\left(\frac{\rho_0}{\rho}\right)\cos\Delta\phi\right)^{\frac{1}{2}}$ can be expressed in terms of an infinite series, i.e., $R = \rho\left(1 - \cos\Delta\phi\left(\frac{\rho_0}{\rho}\right) + \frac{1}{2}(\sin\Delta\phi)^2\left(\frac{\rho_0}{\rho}\right)^2 + \frac{1}{2}\cos\Delta\phi\,(\sin\Delta\phi)^2\left(\frac{\rho_0}{\rho}\right)^3 + \cdots\right)$. (6-b)

In the Far-Field region, which is defined by $\rho \geq \frac{2D^2}{\lambda}$ and $D \equiv 2\rho_0$, the R in the phase term is approximated using the first two terms of the above series, i.e., $R \approx \rho - \rho_0\cos\Delta\phi$ which in turn can be re-written as $R = \rho - \boldsymbol{a}_\rho \cdot \boldsymbol{\rho}_0$. Furthermore, the R in the amplitude term is approximated using only the first term of the above series, i.e., $R \approx \rho$.

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$$E_z = -\left(\eta\sqrt{\frac{k}{8\pi}}e^{j\left(k\boldsymbol{a}_\rho\cdot\boldsymbol{\rho}_0+\frac{\pi}{4}\right)}\right)I_0\frac{1}{\sqrt{\rho}}e^{-jk\rho}$$

Since H_ρ contains the expression $\frac{\sin\Delta\phi}{\left(\frac{\rho}{\rho_0}\right)-\cos\Delta\phi}$, and the quantity $\frac{\rho}{\rho_0}$ becomes infinitely large in the Far-Field region, Therefore, $H_\rho \approx 0$.

(5), (6) →

$$\begin{aligned} H_\phi &= -j\frac{k}{4}I_0\frac{H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|)}{|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}(\rho-\rho_0\cos\Delta\phi) \\ &\approx -j\frac{k}{4}I_0H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|) \\ &= -\left(\sqrt{\frac{k}{8\pi}}e^{j\left(k\boldsymbol{a}_\rho\cdot\boldsymbol{\rho}_0+\frac{3\pi}{4}\right)}\right)I_0\frac{1}{\sqrt{\rho}}e^{-jk\rho} \end{aligned}$$

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15- Propagation in Magnetic Media

Table 41. Plane Wave Propagation in Ferrite Medium in the Direction of Magnetic Bias

$$\nabla \cdot \boldsymbol{D} = \rho_e \tag{1-a}$$
$$\nabla \times \boldsymbol{E} = -\boldsymbol{M} - j\omega\boldsymbol{\mu}\boldsymbol{H} \tag{1-b}$$
$$\nabla \cdot \boldsymbol{B} = \rho_m \tag{1-c}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E} \tag{1-d}$$

Properties of the Medium

Source less

$$\rho_e = 0, \rho_m = 0, \boldsymbol{M} = \boldsymbol{0}, \boldsymbol{J} = \boldsymbol{0} \tag{2}$$

Direction of Magnetic Bias

$$\boldsymbol{H}_{bias} = H_{bias}\boldsymbol{a}_z \tag{3}$$

Permeability Tensor

$$\boldsymbol{\mu} = \begin{bmatrix} \mu & j\kappa & 0 \\ -j\kappa & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix} \tag{4}$$

Direction of Propagation

$$\boldsymbol{a}_{r,p} = \boldsymbol{a}_z \tag{5}$$

Conditions for Uniform Plane Wave

(5) →

$$\frac{\partial}{\partial x} = 0 \tag{6-a}$$
$$\frac{\partial}{\partial y} = 0 \tag{6-b}$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-j\beta \boldsymbol{a}_{r,p} \cdot \boldsymbol{r}} \tag{7-a}$$

$$\boldsymbol{E}_0 = E_{0x}\boldsymbol{a}_x + E_{0y}\boldsymbol{a}_y + E_{0z}\boldsymbol{a}_z \tag{7-b}$$

$$\boldsymbol{H} = \boldsymbol{H}_0 e^{-j\beta \boldsymbol{a}_{r,p} \cdot \boldsymbol{r}} \tag{8-a}$$

$$\boldsymbol{H}_0 = H_{0x}\boldsymbol{a}_x + H_{0y}\boldsymbol{a}_y + H_{0z}\boldsymbol{a}_z \tag{8-b}$$

(1-b), (2), (6), (7), (8) →

$$E_{0x} = -j\frac{\omega}{\beta}\kappa H_{0x} + \mu\frac{\omega}{\beta}H_{0y} \tag{9-a}$$
$$E_{0y} = -\mu\frac{\omega}{\beta}H_{0x} - j\frac{\omega}{\beta}\kappa H_{0y} \tag{9-b}$$
$$H_{0z} = 0 \tag{9-c}$$

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(1-d), (2), (6), (7), (8) →

$$H_{0x} = -\frac{\omega\varepsilon}{\beta} E_{0y} \quad (10\text{-a})$$

$$H_{0y} = \frac{\omega\varepsilon}{\beta} E_{0x} \quad (10\text{-b})$$

$$E_{0z} = 0 \quad (10\text{-c})$$

(9-a), (10-a), (10-b) →

$$(\omega^2\mu\varepsilon - \beta^2)E_{0x} + (j\omega^2\varepsilon\kappa)E_{0y} = 0 \quad (11\text{-a})$$

(9-b), (10-a), (10-b) →

$$(-j\omega^2\varepsilon\kappa)E_{0x} + (\omega^2\mu\varepsilon - \beta^2)E_{0y} = 0 \quad (11\text{-b})$$

Non-trivial Solutions

(11) →

$$\begin{vmatrix} \omega^2\mu\varepsilon - \beta^2 & j\omega^2\varepsilon\kappa \\ -j\omega^2\varepsilon\kappa & \omega^2\mu\varepsilon - \beta^2 \end{vmatrix} = 0 \quad (12)$$

(12) →

$$\beta_1 = \omega\sqrt{(\mu + \kappa)\varepsilon} \quad (13\text{-a})$$

$$\beta_2 = \omega\sqrt{(\mu - \kappa)\varepsilon} \quad (13\text{-b})$$

(11-a), (13-a) →

$$E_{0x} = jE_{0y} \quad (14)$$

(11-a), (13-b) →

$$E_{0x} = -jE_{0y} \quad (15)$$

RHCP Uniform Plane Wave

$$\boldsymbol{E}^{RHCP} = \boldsymbol{E}_0^{RHCP} e^{-j\beta^{RHCP}\boldsymbol{a}_{r,p}\cdot\boldsymbol{r}}$$

(7), (10-c), (13-a), (14) →

$$\boldsymbol{E}_0^{RHCP} = E_0(j\boldsymbol{a}_x + \boldsymbol{a}_y) \quad (16\text{-a})$$

$$\beta^{RHCP} = \omega\sqrt{\mu_{eff}^{RHCP}\varepsilon} \qquad \mu_{eff}^{RHCP} \equiv \mu + \kappa$$

(8), (9-c), (10-a), (10-b) →

$$\boldsymbol{H}^{RHCP} = \frac{1}{\eta^{RHCP}}\boldsymbol{a}_{r,p} \times \boldsymbol{E}^{RHCP} \qquad \eta^{RHCP} \equiv \sqrt{\frac{\mu_{eff}^{RHCP}}{\varepsilon}} \quad (16\text{-b})$$

LHCP Uniform Plane Wave

$$\boldsymbol{E}^{LHCP} = \boldsymbol{E}_0^{LHCP} e^{-j\beta^{LHCP}\boldsymbol{a}_{r,p}\cdot\boldsymbol{r}}$$

(7), (10-c), (13-b), (15) →

$$\boldsymbol{E}_0^{LHCP} = -E_0(j\boldsymbol{a}_x - \boldsymbol{a}_y) \quad (17\text{-a})$$

$$\beta^{LHCP} = \omega\sqrt{\mu_{eff}^{LHCP}\varepsilon} \qquad \mu_{eff}^{LHCP} \equiv \mu - \kappa$$

(8), (9-c), (10-a), (10-b)

$$\boldsymbol{H}^{LHCP} = \frac{1}{\eta^{LHCP}}\boldsymbol{a}_{r,p} \times \boldsymbol{E}^{LHCP} \qquad \eta^{LHCP} \equiv \sqrt{\frac{\mu_{eff}^{LHCP}}{\varepsilon}} \quad (17\text{-b})$$

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Faraday Rotation

(16-a), (17-a) $\rightarrow$

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^{RHCP} + \boldsymbol{E}^{LHCP} \\ &= \boldsymbol{E}_0^{RHCP} e^{-j\beta^{RHCP}\boldsymbol{a}_{r,p}\cdot\boldsymbol{r}} + \boldsymbol{E}_0^{LHCP} e^{-j\beta^{LHCP}\boldsymbol{a}_{r,p}\cdot\boldsymbol{r}} \\ &= E_0\left[(j\boldsymbol{a}_x + \boldsymbol{a}_y)e^{-j\beta^{RHCP}z} - (j\boldsymbol{a}_x - \boldsymbol{a}_y)e^{-j\beta^{LHCP}z}\right] \end{aligned} \tag{18}$$

$$\begin{aligned} \beta^{RHCP} &= \frac{1}{2}(\beta^{RHCP} + \beta^{LHCP}) + \frac{1}{2}(\beta^{RHCP} - \beta^{LHCP}) \\ &= \frac{1}{2}\beta^{\Sigma} + \frac{1}{2}\beta^{\Delta} \end{aligned} \tag{19-a}$$

$$\begin{aligned} \beta^{LHCP} &= \frac{1}{2}(\beta^{RHCP} + \beta^{LHCP}) - \frac{1}{2}(\beta^{RHCP} - \beta^{LHCP}) \\ &= \frac{1}{2}\beta^{\Sigma} - \frac{1}{2}\beta^{\Delta} \end{aligned} \tag{19-b}$$

(18), (19) $\rightarrow$

$$\begin{aligned} \boldsymbol{E} &= E_0\left[(j\boldsymbol{a}_x + \boldsymbol{a}_y)e^{-j\frac{1}{2}\beta^{\Delta}z} - (j\boldsymbol{a}_x - \boldsymbol{a}_y)e^{j\frac{1}{2}\beta^{\Delta}z}\right]e^{-j\frac{1}{2}\beta^{\Sigma}z} \\ &= 2E_0\left[\sin\left(\frac{1}{2}\beta^{\Delta}z\right)\boldsymbol{a}_x + \cos\left(\frac{1}{2}\beta^{\Delta}z\right)\boldsymbol{a}_y\right]e^{-j\frac{1}{2}\beta^{\Sigma}z} \end{aligned}$$

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16- Electrostatics and Magnetostatics

Table 42. Derivation of the Complete Static Solutions

		$\boldsymbol{\nabla \cdot D} = \rho$	(1-a)
		$\boldsymbol{\nabla \times E} = \boldsymbol{0}$	(1-b)
		$\boldsymbol{\nabla \cdot B} = 0$	(1-c)
		$\boldsymbol{\nabla \times H} = \boldsymbol{J}$	(1-d)
		$\boldsymbol{\nabla \cdot (\nabla \times A)} = 0$	VI-1
		$\boldsymbol{\nabla \times (\nabla}\psi) = \boldsymbol{0}$	VI-2
		$-\boldsymbol{\nabla \times \nabla \times A} + \boldsymbol{\nabla(\nabla \cdot A)} = \boldsymbol{\nabla^2 A}$	VI-3
		$\boldsymbol{\nabla}(ab) = a\boldsymbol{\nabla}(b) + b\boldsymbol{\nabla}(a)$	VI-4
		$\boldsymbol{\nabla \times} (a\boldsymbol{A}) = a(\boldsymbol{\nabla \times A}) + (\boldsymbol{\nabla}a) \times \boldsymbol{A}$	VI-5
		Useful Math	
		$\boldsymbol{\nabla}\left(\frac{1}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}\right) = -\frac{\boldsymbol{r} - \boldsymbol{r}'}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert^3}$	UI-1
		Electrostatic Fields	
(1-b), VI-2	→	$\boldsymbol{E} = -\boldsymbol{\nabla}V$	(2)
(1-a), (2)	→	$\nabla^2 V = -\frac{1}{\varepsilon}\rho$	(3)
		Solution to the Inhomogeneous Laplace Eq.	
		$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\iiint_{V'} \rho(\boldsymbol{r}')\frac{1}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert}dv'$	(4)
(2), (4), VI-4, UI-1	→	$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\iiint_{V'} \rho(\boldsymbol{r}')\frac{\boldsymbol{r} - \boldsymbol{r}'}{\lvert \boldsymbol{r} - \boldsymbol{r}' \rvert^3}dv'$	
		Magnetostatic Fields	
(1-c), VI-1	→	$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla \times A}$	(5)
(1-d), (5), VI-3	→	$\boldsymbol{\nabla^2 A} - \boldsymbol{\nabla(\nabla \cdot A)} = -\mu\boldsymbol{J}$	(6)
		Coulomb Gauge	
		$\boldsymbol{\nabla \cdot A} = 0$	(7)

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(6), (7) → $\boldsymbol{\nabla^2 A = -\mu J}$

Solution to the Inhomogeneous Laplace Eq.

$$\boldsymbol{A(r)} = \frac{\mu}{4\pi}\iiint_{V'} \boldsymbol{J(r')}\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|}dv' \tag{8}$$

(5), (8), VI-5, UI-1 → $$\boldsymbol{H} = \frac{1}{4\pi}\iiint_{V'} \boldsymbol{J(r')}\times\frac{\boldsymbol{r}-\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|^3}dv'$$

Table 43. Static Fields Generated by Hypothetical Sources

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon}\iiint_{V'} \rho(\boldsymbol{r'})\frac{\boldsymbol{r}-\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|^3}dv' \tag{1-a}$$

$$\boldsymbol{H} = \frac{1}{4\pi}\iiint_{V'} \boldsymbol{J(r')}\times\frac{\boldsymbol{r}-\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|^3}dv' \tag{1-b}$$

Useful Math

$$\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}}dx = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} \tag{UR-1}$$

$$\int \frac{x}{(x^2+a^2)^{\frac{3}{2}}}dx = -\frac{1}{\sqrt{x^2+a^2}} \tag{UR-2}$$

$$\int \cos(\phi-\phi')\cos\phi'\,d\phi' = \frac{1}{2}\phi'\cos\phi - \frac{1}{4}\sin(\phi-2\phi') \tag{UR-3}$$

$$\int \cos(\phi-\phi')\sin\phi'\,d\phi' = \frac{1}{2}\phi'\sin\phi - \frac{1}{4}\cos(\phi-2\phi') \tag{UR-4}$$

$$\int (\cos(\phi-\phi'))^2 d\phi' = \frac{1}{2}\phi' - \frac{1}{2}\phi - \frac{1}{4}\sin(2\phi-2\phi') \tag{UR-5}$$

Electric Sources

Impulse of Charge Density (Point Charge)

$$\rho(\boldsymbol{r}) = q\delta(x)\delta(y)\delta(z) \tag{2-a}$$

$$\boldsymbol{r} = \boldsymbol{0} \tag{2-b}$$

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$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\iiint_{V'}[q\delta(x')\delta(y')\delta(z')]\frac{\boldsymbol{r}-\boldsymbol{0}}{|\boldsymbol{r}-\boldsymbol{0}|^3}dv'$$

(1-a), (2) $\rightarrow$

$$= \frac{1}{4\pi\varepsilon}q\frac{1}{r^2}\boldsymbol{a}_r\left[\int\delta(x')dx'\right]\left[\int\delta(y')dy'\right]\left[\int\delta(z')dz'\right]$$

$$= \frac{1}{4\pi\varepsilon}q\frac{1}{r^2}\boldsymbol{a}_r$$

Infinite Line of Charge Density (Line Charge)

$$\rho(\boldsymbol{r}) = \rho_0\delta(\rho)\left(\frac{1}{\rho}\delta(\phi)\right) \tag{3-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{3-b}$$

Observation point lies on the $z = 0$ plane. (3-c)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\iiint_{V'}\left[\rho_0\delta(\rho')\left(\frac{1}{\rho'}\delta(\phi')\right)\right]\frac{\boldsymbol{r}-z'\boldsymbol{a}_z}{|\boldsymbol{r}-z'\boldsymbol{a}_z|^3}dv'$$

$$= \frac{1}{4\pi\varepsilon}\iiint_{V'}\left[\rho_0\delta(\rho')\left(\frac{1}{\rho'}\delta(\phi')\right)\right]\frac{\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'(\rho'd\phi')d\rho'$$

$$= \frac{1}{4\pi\varepsilon}\rho_0\rho\left(\iiint_{V'}\delta(\rho')\delta(\phi')\frac{1}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'd\phi'd\rho'\right)\boldsymbol{a}_\rho$$

$$- \frac{1}{4\pi\varepsilon}\rho_0\left(\iiint_{V'}\delta(\rho')\delta(\phi')\frac{z'}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'd\phi'd\rho'\right)\boldsymbol{a}_z$$

(1-a), (3) $\rightarrow$

$$= \frac{1}{4\pi\varepsilon}\rho_0\rho\left(\left[\int\delta(\rho')d\rho'\right]\left[\int\delta(\phi')d\phi'\right]\left[\int_{-\infty}^{+\infty}\frac{1}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'\right]\right)\boldsymbol{a}_\rho \tag{4}$$

$$- \frac{1}{4\pi\varepsilon}\rho_0\left(\left[\int\delta(\rho')d\rho'\right]\left[\int\delta(\phi')d\phi'\right]\left[\int_{-\infty}^{+\infty}\frac{z'}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'\right]\right)\boldsymbol{a}_z$$

$$= \frac{1}{4\pi\varepsilon}\rho_0\rho\left(\int_{-\infty}^{+\infty}\frac{1}{\left(\rho^2+z'^2\right)^{\frac{3}{2}}}dz'\right)\boldsymbol{a}_\rho$$

$$- \frac{1}{4\pi\varepsilon}\rho_0\left(\int_{-\infty}^{+\infty}\frac{z'}{\left(\rho^2+z'^2\right)^{\frac{3}{2}}}dz'\right)\boldsymbol{a}_z$$

(4), UR-1, UR-2 $\rightarrow$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{2\pi\varepsilon}\rho_0\frac{1}{\rho}\boldsymbol{a}_\rho$$

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Electric Dipole

$$\rho(\boldsymbol{r}) = q\delta(x)\delta(y)\delta\left(z-\frac{d}{2}\right) - q\delta(x)\delta(y)\delta\left(z+\frac{d}{2}\right) \tag{5}$$

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon}\iiint_{V'}\left[q\delta(x')\delta(y')\delta\left(z'-\frac{d}{2}\right)\right]\frac{\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z\right|^3}dv' \\
&- \frac{1}{4\pi\varepsilon}\iiint_{V'}\left[q\delta(x')\delta(y')\delta\left(z'+\frac{d}{2}\right)\right]\frac{\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z\right|^3}dv' \\
\text{(1-a), (5)} \quad\rightarrow\quad &= \frac{1}{4\pi\varepsilon}q\frac{\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z\right|^3}\left[\int\delta(x')dx'\right]\left[\int\delta(y')dy'\right]\left[\int\delta(z')dz'\right] \qquad (6)\\
&- \frac{1}{4\pi\varepsilon}q\frac{\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z\right|^3}\left[\int\delta(x')dx'\right]\left[\int\delta(y')dy'\right]\left[\int\delta(z')dz'\right] \\
&= \frac{1}{4\pi\varepsilon}q\left(\frac{\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}-\frac{d}{2}\boldsymbol{a}_z\right|^3}-\frac{\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z}{\left|\boldsymbol{r}+\frac{d}{2}\boldsymbol{a}_z\right|^3}\right)
\end{aligned}$$

The Large Distance Approximation

$$\begin{aligned}
|\boldsymbol{r}\pm z_0\boldsymbol{a}_z|^{-3} &= [(\boldsymbol{r}\pm z_0\boldsymbol{a}_z)\cdot(\boldsymbol{r}\pm z_0\boldsymbol{a}_z)]^{-\frac{3}{2}} \\
&= [|\boldsymbol{r}|^2+z_0^2\pm 2z_0\boldsymbol{a}_z\cdot\boldsymbol{r}]^{-\frac{3}{2}} \\
&= |\boldsymbol{r}|^{-3}\left[1+\left(\frac{z_0}{|\boldsymbol{r}|}\right)^2\pm 2\left(\frac{z_0}{|\boldsymbol{r}|}\right)\boldsymbol{a}_z\cdot\boldsymbol{a}_r\right]^{-\frac{3}{2}}
\end{aligned} \tag{7-a}$$

At large distances from the source, i.e., $\frac{z_0}{|\boldsymbol{r}|}\ll 1$, the expression in (7-a) can be approximated as $|\boldsymbol{r}|^{-3}\left[1\pm 2\left(\frac{z_0}{|\boldsymbol{r}|}\right)\boldsymbol{a}_z\cdot\boldsymbol{a}_r\right]^{-\frac{3}{2}}$. This expression can be further simplified by keeping only the first two terms of its Tylor expansion. (7-b)

(7-a), (7-b) $\rightarrow$ $$\frac{1}{|\boldsymbol{r}\pm z_0\boldsymbol{a}_z|^3}\approx\frac{1}{|\boldsymbol{r}|^3}\left(1\mp 3\left(\frac{z_0}{|\boldsymbol{r}|}\right)\boldsymbol{a}_z\cdot\boldsymbol{a}_r\right) \tag{7-c}$$

(6), (7) $\rightarrow$ $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\frac{1}{r^3}[3(\boldsymbol{P}\cdot\boldsymbol{a}_r)\boldsymbol{a}_r-\boldsymbol{P}]$ in which $\boldsymbol{P}\equiv qd\boldsymbol{a}_z$

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Magnetic Sources

Infinite Line of Current Density

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho)\left(\frac{1}{\rho}\delta(\phi)\right)\boldsymbol{a}_z \tag{8-a}$$

$$\boldsymbol{r} = z\boldsymbol{a}_z \tag{8-b}$$

Observation point lies on the $z = 0$ plane. (8-c)

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{r}) &= \frac{1}{4\pi}\iiint_{V'}\left[I_0\delta(\rho')\left(\frac{1}{\rho'}\delta(\phi')\right)\boldsymbol{a}_z\right]\times\frac{\boldsymbol{r}-z'\boldsymbol{a}_z}{|\boldsymbol{r}-z'\boldsymbol{a}_z|^3}dv' \\
&= \frac{1}{4\pi}\iiint_{V'}\left[I_0\delta(\rho')\left(\frac{1}{\rho'}\delta(\phi')\right)\boldsymbol{a}_z\right]\times\frac{\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'(\rho'd\phi')d\rho' \\
\text{(1-b), (8)} \rightarrow \quad &= \frac{1}{4\pi}I_0\rho\left(\iiint_{V'}\delta(\rho')\delta(\phi')\frac{1}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'd\phi'd\rho'\right)\boldsymbol{a}_\phi \qquad (9) \\
&= \frac{1}{4\pi}I_0\rho\left(\left[\int\delta(\rho')d\rho'\right]\left[\int\delta(\phi')d\phi'\right]\left[\int_{-\infty}^{+\infty}\frac{1}{|\rho\boldsymbol{a}_\rho-z'\boldsymbol{a}_z|^3}dz'\right]\right)\boldsymbol{a}_\phi \\
&= \frac{1}{4\pi}I_0\rho\left(\int_{-\infty}^{+\infty}\frac{1}{\left(\rho^2+z'^2\right)^{\frac{3}{2}}}dz'\right)\boldsymbol{a}_\phi
\end{aligned}$$

$$\text{(9), UR-1} \rightarrow \quad \boldsymbol{H}(\boldsymbol{r}) = \frac{1}{2\pi}I_0\frac{1}{\rho}\boldsymbol{a}_\phi$$

Magnetic Dipole

$$\boldsymbol{J}(\boldsymbol{r}) = I_0\delta(\rho-\rho_0)\delta(z)\boldsymbol{a}_\phi \tag{10-a}$$

$$\boldsymbol{r} = \rho_0\boldsymbol{a}_\rho \tag{10-b}$$

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{r}) &= \frac{1}{4\pi}\iiint_{V'}\left[I_0\delta(\rho'-\rho_0)\delta(z')\boldsymbol{a}_{\phi'}\right]\times\frac{\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dv' \\
&= \frac{1}{4\pi}I_0\iiint_{V'}\left[\delta(\rho'-\rho_0)\delta(z')\boldsymbol{a}_{\phi'}\right]\times\frac{\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dz'(\rho'd\phi')d\rho' \\
\text{(1-b), (10)} \rightarrow \quad &= \frac{1}{4\pi}I_0\iiint_{V'}\delta(\rho'-\rho_0)\rho'\delta(z')\frac{\boldsymbol{a}_{\phi'}\times\boldsymbol{r}}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dz'd\phi'd\rho' \qquad (11) \\
&- \frac{1}{4\pi}I_0\rho_0\iiint_{V'}\delta(\rho'-\rho_0)\rho'\delta(z')\frac{\boldsymbol{a}_{\phi'}\times\boldsymbol{a}_{\rho'}}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dz'd\phi'd\rho' \\
&= \frac{1}{4\pi}I_0\iiint_{V'}\delta(\rho'-\rho_0)\rho'\delta(z')\frac{\boldsymbol{a}_{\phi'}\times\boldsymbol{r}}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dz'd\phi'd\rho' \\
&+ \frac{1}{4\pi}I_0\rho_0\left(\iiint_{V'}\delta(\rho'-\rho_0)\rho'\delta(z')\frac{1}{|\boldsymbol{r}-\rho_0\boldsymbol{a}_{\rho'}|^3}dz'd\phi'd\rho'\right)\boldsymbol{a}_z
\end{aligned}$$

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$$\boldsymbol{a}_{\phi'} \times \boldsymbol{a}_r = \left(-\sin\phi'\,\boldsymbol{a}_x + \cos\phi'\,\boldsymbol{a}_y\right) \times \left(\sin\theta\cos\phi\,\boldsymbol{a}_x + \sin\theta\sin\phi\,\boldsymbol{a}_y + \cos\theta\,\boldsymbol{a}_z\right)$$
$$= \cos\theta\cos\phi'\,\boldsymbol{a}_x + \cos\theta\sin\phi'\,\boldsymbol{a}_y - \sin\theta\cos\Delta\phi\,\boldsymbol{a}_z \quad \text{TI-1}$$
in which $\Delta\phi \equiv \phi - \phi'$

The Large Distance Approximation

$$\left|\boldsymbol{r} - \rho_0\boldsymbol{a}_{\rho'}\right|^{-3} = |\boldsymbol{r}|^{-3}\left|\boldsymbol{a}_r - \left(\frac{\rho_0}{|\boldsymbol{r}|}\right)\boldsymbol{a}_{\rho'}\right|^{-3}$$
$$= |\boldsymbol{r}|^{-3}\left(1 + \left(\frac{\rho_0}{|\boldsymbol{r}|}\right)^2 - 2\left(\frac{\rho_0}{|\boldsymbol{r}|}\right)\sin\theta\cos\Delta\phi\right)^{-\frac{3}{2}} \quad (12)$$
$$\approx |\boldsymbol{r}|^{-3}\left(1 + 3\left(\frac{\rho_0}{|\boldsymbol{r}|}\right)\sin\theta\cos\Delta\phi\right)$$

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi}I_0\frac{1}{r^3}\cos\theta\left(\left[\iint \delta(\rho'-\rho_0)\rho'd\rho'\right]\left[\int \delta(z')dz'\right]\left[\int_0^{2\pi}(r + 3\rho_0\sin\theta\cos\Delta\phi)\cos\phi'\,d\phi'\right]\right)\boldsymbol{a}_x$$
$$+ \frac{1}{4\pi}I_0\frac{1}{r^3}\cos\theta\left(\left[\iint \delta(\rho'-\rho_0)\rho'd\rho'\right]\left[\int \delta(z')dz'\right]\left[\int_0^{2\pi}(r + 3\rho_0\sin\theta\cos\Delta\phi)\sin\phi'\,d\phi'\right]\right)\boldsymbol{a}_y$$
$$- \frac{1}{4\pi}I_0\frac{1}{r^3}\sin\theta\left(\left[\iint \delta(\rho'-\rho_0)\rho'd\rho'\right]\left[\int \delta(z')dz'\right]\left[\int_0^{2\pi}(r + 3\rho_0\sin\theta\cos\Delta\phi)\cos\Delta\phi\,d\phi'\right]\right)\boldsymbol{a}_z$$
$$+ \frac{1}{4\pi}I_0\rho_0\frac{1}{r^3}\left(\left[\iint \delta(\rho'-\rho_0)\rho'd\rho'\right]\left[\int \delta(z')dz'\right]\left[\int_0^{2\pi}\left(1 + 3\frac{\rho_0}{r}\sin\theta\cos\Delta\phi\right)d\phi'\right]\right)\boldsymbol{a}_z$$

(11), TI-1, (12) →
$$= \frac{1}{4\pi}I_0\rho_0\frac{1}{r^3}\cos\theta\left(3\rho_0\sin\theta\int_0^{2\pi}\cos\Delta\phi\cos\phi'\,d\phi'\right)\boldsymbol{a}_x$$
$$+ \frac{1}{4\pi}I_0\rho_0\frac{1}{r^3}\cos\theta\left(3\rho_0\sin\theta\int_0^{2\pi}\cos\Delta\phi\sin\phi'\,d\phi'\right)\boldsymbol{a}_y$$
$$- \frac{1}{4\pi}I_0\rho_0\frac{1}{r^3}\sin\theta\left(3\rho_0\sin\theta\int_0^{2\pi}(\cos\Delta\phi)^2 d\phi'\right)\boldsymbol{a}_z$$
$$+ \frac{1}{4\pi r^3}2\left((\pi\rho_0^2)I_0\right)\boldsymbol{a}_z \quad (13)$$

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi r^3}\left((\pi\rho_0^2)I_0\right)3\cos\theta\sin\theta\cos\phi\,\boldsymbol{a}_x$$
$$+ \frac{1}{4\pi r^3}\left((\pi\rho_0^2)I_0\right)3\cos\theta\sin\theta\sin\phi\,\boldsymbol{a}_y$$
(13), UR-3, UR-4, UR-5 →
$$- \frac{1}{4\pi r^3}\left((\pi\rho_0^2)I_0\right)3(\sin\theta)^2\boldsymbol{a}_z \quad (14)$$
$$+ \frac{1}{4\pi r^3}2\left((\pi\rho_0^2)I_0\right)\boldsymbol{a}_z$$

(14) → $\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi}\frac{1}{r^3}[3(\boldsymbol{M}\cdot\boldsymbol{a}_r)\boldsymbol{a}_r - \boldsymbol{M}]$ in which $\boldsymbol{M} \equiv \left((\pi\rho_0^2)I_0\right)\boldsymbol{a}_z$

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17- Useful Math

Vector Calculus

Coordinate Systems

- Rectangular Coordinates

$u_1 = x,\ u_2 = y,\ u_3 = z$

$h_1 = 1,\ h_2 = 1,\ h_3 = 1$

$\boldsymbol{a}_x \cdot \boldsymbol{a}_x = 1,\ \boldsymbol{a}_x \cdot \boldsymbol{a}_y = 0,\ \boldsymbol{a}_x \cdot \boldsymbol{a}_z = 0$

$\boldsymbol{a}_y \cdot \boldsymbol{a}_x = 0,\ \boldsymbol{a}_y \cdot \boldsymbol{a}_y = 1,\ \boldsymbol{a}_y \cdot \boldsymbol{a}_z = 0$

$\boldsymbol{a}_z \cdot \boldsymbol{a}_x = 0,\ \boldsymbol{a}_z \cdot \boldsymbol{a}_y = 0,\ \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$

- Cylindrical Coordinates

$u_1 = \rho,\ u_2 = \phi,\ u_3 = z$

$h_1 = 1,\ h_2 = \rho,\ h_3 = 1$

$\boldsymbol{a}_\rho \cdot \boldsymbol{a}_x = \cos\phi,\ \boldsymbol{a}_\rho \cdot \boldsymbol{a}_y = \sin\phi,\ \boldsymbol{a}_\rho \cdot \boldsymbol{a}_z = 0$

$\boldsymbol{a}_\phi \cdot \boldsymbol{a}_x = -\sin\phi,\ \boldsymbol{a}_\phi \cdot \boldsymbol{a}_y = \cos\phi,\ \boldsymbol{a}_\phi \cdot \boldsymbol{a}_z = 0$

$\boldsymbol{a}_z \cdot \boldsymbol{a}_x = 0,\ \boldsymbol{a}_z \cdot \boldsymbol{a}_y = 0,\ \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$

- Spherical Coordinates

$u_1 = r,\ u_2 = \theta,\ u_3 = \phi$

$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

$\boldsymbol{a}_r \cdot \boldsymbol{a}_x = \sin\theta\cos\phi,\ \boldsymbol{a}_r \cdot \boldsymbol{a}_y = \sin\theta\sin\phi,\ \boldsymbol{a}_r \cdot \boldsymbol{a}_z = \cos\theta$

$\boldsymbol{a}_\theta \cdot \boldsymbol{a}_x = \cos\theta\cos\phi,\ \boldsymbol{a}_\theta \cdot \boldsymbol{a}_y = \cos\theta\sin\phi,\ \boldsymbol{a}_\theta \cdot \boldsymbol{a}_z = -\sin\theta$

$\boldsymbol{a}_\phi \cdot \boldsymbol{a}_x = -\sin\phi,\ \boldsymbol{a}_\phi \cdot \boldsymbol{a}_y = \cos\phi,\ \boldsymbol{a}_\phi \cdot \boldsymbol{a}_z = 0$

The Chain Rule

$$\begin{bmatrix} \frac{\partial\psi}{\partial u_1} \\ \frac{\partial\psi}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial\psi}{\partial v_1} \\ \frac{\partial\psi}{\partial v_2} \end{bmatrix}^T \begin{bmatrix} \frac{\partial v_1}{\partial u_1} & \frac{\partial v_1}{\partial u_2} \\ \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial u_2} \end{bmatrix}$$

- Cylindrical Coordinates

$$\begin{aligned} \begin{bmatrix} \frac{\partial\psi}{\partial\rho} \\ \frac{\partial\psi}{\partial\phi} \end{bmatrix} &= \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix}^T \begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix}^T \begin{bmatrix} \frac{\partial\rho}{\partial x} & \frac{\partial\phi}{\partial x} \\ \frac{\partial\rho}{\partial y} & \frac{\partial\phi}{\partial y} \end{bmatrix}^{-1} \end{aligned}$$

Useful Vector Identities

$$\begin{aligned} a(\boldsymbol{A}\times\boldsymbol{B}) &= (a\boldsymbol{A})\times\boldsymbol{B} \\ &= \boldsymbol{A}\times(a\boldsymbol{B}) \\ \boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \\ \boldsymbol{\nabla}(ab) &= a(\boldsymbol{\nabla}b) + b(\boldsymbol{\nabla}a) \\ \boldsymbol{\nabla}\cdot(a\boldsymbol{A}) &= a\boldsymbol{\nabla}\cdot\boldsymbol{A} + (\boldsymbol{\nabla}a)\cdot\boldsymbol{A} \\ \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{B} - (\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\boldsymbol{A} \\ \boldsymbol{\nabla}\times(a\boldsymbol{A}) &= a\boldsymbol{\nabla}\times\boldsymbol{A} + (\boldsymbol{\nabla}a)\times\boldsymbol{A} \\ \boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{\nabla}\cdot\boldsymbol{A})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \end{aligned}$$

First-Order Derivatives

$$\boldsymbol{\nabla}\psi = \left(\frac{1}{h_1}\right)\frac{\partial\psi}{\partial u_1}\boldsymbol{a}_{u_1} + \left(\frac{1}{h_2}\right)\frac{\partial\psi}{\partial u_2}\boldsymbol{a}_{u_2} + \left(\frac{1}{h_3}\right)\frac{\partial\psi}{\partial u_3}\boldsymbol{a}_{u_3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_{u_1}) + \frac{\partial}{\partial u_2}(h_1h_3A_{u_2}) + \frac{\partial}{\partial u_3}(h_1h_2A_{u_3})\right]$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{a}_{u_1} & h_2\boldsymbol{a}_{u_2} & h_3\boldsymbol{a}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_{u_1} & h_2A_{u_2} & h_3A_{u_3} \end{vmatrix}$$

Second-Order Derivatives

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) &= 0 \\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) &= \boldsymbol{0} \\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi) &\equiv \nabla^2\psi \\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} &\equiv \boldsymbol{\nabla}^2\boldsymbol{A} \end{aligned}$$

The Fundamental Theorems

- Divergence Theorem

$$\oiint_S \boldsymbol{A}\cdot d\boldsymbol{S} = \iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{A}\,dV$$

- Stock's Theorem

$$\oint_C \boldsymbol{A}\cdot d\boldsymbol{l} = \iint_S (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot d\boldsymbol{S}$$

Partial Differential Equations (PDEs): The Scalar Laplacian Operator

- Rectangular Coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

- Cylindrical Coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Partial Differential Equations (PDEs): The Vector Laplacian Operator

- Rectangular Coordinates

$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_x = \nabla^2A_x$

$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_y = \nabla^2A_y$

$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_z = \nabla^2A_z$

- Cylindrical Coordinates

$$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_\rho = \nabla^2A_\rho + \left(-\frac{1}{\rho^2}A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi}\right)$$

$$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_\phi = \nabla^2A_\phi + \left(-\frac{1}{\rho^2}A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi}\right)$$

$(\boldsymbol{\nabla}^2\boldsymbol{A})\cdot\boldsymbol{a}_z = \nabla^2A_z$

Partial Differential Equations (PDEs): Solution to the Inhomogeneous Laplace Equation

- $\nabla^2\psi = f$
- Rectangular Coordinates

$$\psi(\boldsymbol{r}) = -\frac{1}{4\pi}\iiint_{V'} f\left(\boldsymbol{r}'\right)\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

Partial Differential Equations (PDEs): Solution to the Inhomogeneous Wave Equation

- $\nabla^2\psi + k^2\psi = f$
- Rectangular Coordinates

$$\psi(\boldsymbol{r}) = -\frac{1}{4\pi}\iiint_{V'} f\left(\boldsymbol{r}'\right)\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

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Partial Differential Equations (PDEs): Solutions to the Homogeneous Laplace Equation (2D)

- $\nabla^2\psi = 0$
- Rectangular Coordinates

$\psi(\boldsymbol{r}) = \Psi_x(x)\,\Psi_y(y)\,,\ \left(-k_x^2\right) + \left(-k_y^2\right) = 0$

	$k_u^2 > 0$	$k_u^2 < 0$	$k_u = 0$
Ψ_u	$A\sin(ku) + B\cos(ku)$	$C\sinh(ku) + D\cosh(ku)$	$Eu + F$

- Cylindrical Coordinates

$\psi(\boldsymbol{\rho}) = \Psi_\rho(\rho)\,\Psi_\phi(\phi)\,,\ \left(k_\rho^2\right) + \left(-k_\phi^2\right) = 0$

	$k_\rho^2 > 0$	$k_\rho = 0$
Ψ_ρ	$A_1 r^m + B_1 r^{-m}$	$C_1\ln(r) + D_1$

	$k_\phi^2 > 0$	$k_\phi = 0$
Ψ_ϕ	$A_2\sin\left(m\phi\right) + B_2\cos\left(m\phi\right)$	$C_2\phi + D_2$

Partial Differential Equations (PDEs): Solutions to the Homogeneous Wave Equation

- $\nabla^2\psi + k^2\psi = 0$
- Rectangular Coordinates

$\psi(\boldsymbol{r}) = \Psi_x(x)\,\Psi_y(y)\,\Psi_z(z)\,,\ \left(-k_x^2\right) + \left(-k_y^2\right) + \left(-k_z^2\right) + k^2 = 0$

	Traveling Wave	**Standing Wave**	**Evanescent Wave**
Ψ_u	$Ae^{-jk_u u} + Be^{jk_u u}$	$C\sin(k_u u) + D\cos(k_u u)$	$E\sinh(k_u u) + F\cosh(k_u u)$

- Cylindrical Coordinates

$\psi(\boldsymbol{r}) = \Psi_\rho(\rho)\,\Psi_\phi(\phi)\,\Psi_z(z)\,,\ \left(-k_\rho^2\right) + \left(-k_z^2\right) + k^2 = 0$

	Traveling Wave	**Standing Wave**	**Evanescent Wave**
Ψ_ρ	$A_1 H_m^{(2)}\left(k_\rho\rho\right) + B_1 H_m^{(1)}\left(k_\rho\rho\right)$	$C_1 J_m\left(k_\rho\rho\right) + D_1 Y_m\left(k_\rho\rho\right)$	$E_1 K_m\left(k_\rho\rho\right) + F_1 I_m\left(k_\rho\rho\right)$
Ψ_ϕ		$C_2\cos\left(m\phi\right) + D_2\sin\left(m\phi\right)$	
Ψ_z	$A_3 e^{-jk_z z} + B_3 e^{jk_z z}$	$C_3\cos(k_z z) + D_3\sin(k_z z)$	

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Convolution, Transforms and Transfer Functions

Convolution

$$f * g(t) \equiv \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

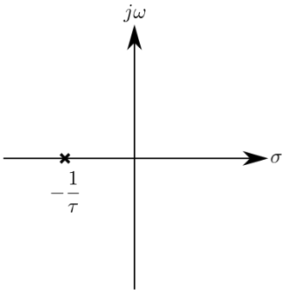
Transforms: The Laplace Transform

$$x(t) = -j\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(S)\, e^{St}\, dS,\ X(S) = \int_{-\infty}^{+\infty} x(t)\, e^{-St}\, dt$$

Transforms: The Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega,\ X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Transfer Functions: First-Order Systems: The Lowpass Filter



- Frequency Domain

$$H(S) = K\frac{1}{S+\frac{1}{\tau}}$$

$$|H(j\omega)| = K\frac{1}{\sqrt{\omega^2+\frac{1}{\tau^2}}}$$

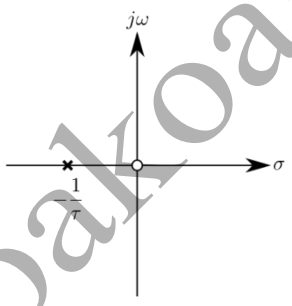
- Time Domain

$$h(t) = Ke^{-\frac{1}{\tau}t}$$

$$s(t) = K\tau\left(1-e^{-\frac{1}{\tau}t}\right)$$

ω	0	$\frac{1}{\tau}$	∞
$\lvert H(j\omega)\rvert$	$K\tau$	$\frac{1}{\sqrt{2}}K\tau$	0

Transfer Functions: First-Order Systems: The Highpass Filter



- Frequency Domain

$$H(S) = K\frac{S}{S+\frac{1}{\tau}}$$

$$|H(j\omega)| = K\frac{\omega}{\sqrt{\omega^2+\frac{1}{\tau^2}}}$$

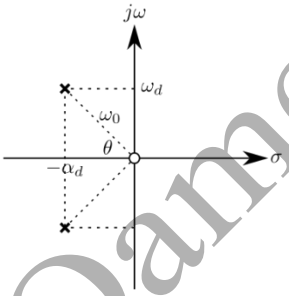
- Time Domain

$$h(t) = K\left(\delta(t)-\frac{1}{\tau}e^{-\frac{1}{\tau}t}\right)$$

$$s(t) = Ke^{-\frac{1}{\tau}t}$$

ω	0	$\frac{1}{\tau}$	∞
$\lvert H(j\omega)\rvert$	0	$\frac{1}{\sqrt{2}}K$	K

Transfer Functions: Second-Order Systems: The Bandpass Filter



- Frequency Domain

$$H(S) = K\frac{S}{S^2+\frac{\omega_0}{Q}S+\omega_0^2}$$

$$|H(j\omega)| = K\frac{\omega}{\sqrt{\omega^4+\omega_0^2\left(\frac{1}{Q^2}-2\right)\omega^2+\omega_0^4}}$$

$$Q \equiv \frac{1}{2\cos\theta},\ \alpha_d = \omega_0\left(\frac{1}{2Q}\right),\ \omega_d = \omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}$$

- Time Domain

$$h(t) = \left(K\frac{\omega_0}{\omega_d}\right)e^{-\alpha_d t}\cos(\omega_d t+\phi),\ \phi \equiv \tan^{-1}\left(\frac{\alpha_d}{\omega_d}\right)$$

$$s(t) = \left(K\frac{1}{\omega_d}\right)e^{-\alpha_d t}\sin(\omega_d t)$$

ω	0	$\omega_0\left(\sqrt{1+\left(\frac{1}{2Q}\right)^2}-\frac{1}{2Q}\right)$	ω_0	$\omega_0\left(\sqrt{1+\left(\frac{1}{2Q}\right)^2}+\frac{1}{2Q}\right)$	∞
$\lvert H(j\omega)\rvert$	0	$\frac{1}{\sqrt{2}}K\frac{Q}{\omega_0}$	$K\frac{Q}{\omega_0}$	$\frac{1}{\sqrt{2}}K\frac{Q}{\omega_0}$	0

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Special Functions, Series Expansion and Trigonometry

Special Functions: The Dirac Delta Function

- $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1,\ \int_{-\infty}^{+\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$
- $\iiint_V \delta(\boldsymbol{r})\,dV = 1$
- $\delta(\boldsymbol{r}-\boldsymbol{r}_0) \equiv \left(\frac{1}{h_1}\delta(u_1-u_{1,0})\right)\left(\frac{1}{h_2}\delta(u_2-u_{2,0})\right)\left(\frac{1}{h_3}\delta(u_3-u_{3,0})\right)$

Special Functions: The Rectangular Function

$$\Pi(t) \equiv \begin{cases} 0, & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2}, & \text{if } |t| = \frac{1}{2} \\ 1, & \text{if } |t| < \frac{1}{2} \end{cases}$$

Special Functions: The Bessel Functions

- In what follows, $\psi_m(x)$ stands for any of $J_m(x)$, $Y_m(x)$, $H_m^{(1)}(x)$ or $H_m^{(2)}(x)$.
- $\psi_m(x) = \frac{x}{2m}\left(\psi_{m-1}(x) + \psi_{m+1}(x)\right)$
- $\psi_{-m}(x) = (-1)^m \psi_m(x)$
- $\frac{d}{dx}\psi_m(x) = \frac{1}{2}\left(\psi_{m-1}(x) - \psi_{m+1}(x)\right)$

Special Functions: The Chebyshev Polynomials

- $\mathrm{T}_n(x) = 2x\mathrm{T}_{n-1}(x) - \mathrm{T}_{n-2}(x)$
- $\mathrm{T}_1(x) = x$

$\mathrm{T}_2(x) = 2x^2 - 1$

$\mathrm{T}_3(x) = 4x^3 - 3x$

$\mathrm{T}_4(x) = 8x^4 - 8x^2 + 1$

$\mathrm{T}_5(x) = 16x^5 - 20x^3 + 5x$

$\mathrm{T}_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

$\mathrm{T}_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

Special Functions: Trigonometric Chebyshev Polynomials

$\mathrm{T}_2(\alpha\cos\theta) = (\alpha^2 - 1) + \alpha^2\cos(2\theta)$

$\mathrm{T}_3(\alpha\cos\theta) = 3\left(\alpha^3 - \alpha\right)\cos(\theta) + \alpha^3\cos(3\theta)$

$\mathrm{T}_4(\alpha\cos\theta) = \left(3\alpha^4 - 4\alpha^2 + 1\right) + 4\left(\alpha^4 - \alpha^2\right)\cos(2\theta) + \alpha^4\cos(4\theta)$

$\mathrm{T}_5(\alpha\cos\theta) = 5\left(2\alpha^5 - 3\alpha^3 + \alpha\right)\cos(\theta) + 5\left(\alpha^5 - \alpha^3\right)\cos(3\theta) + \alpha^5\cos(5\theta)$

$\mathrm{T}_6(\alpha\cos\theta) = \left(10\alpha^6 - 18\alpha^4 + 9\alpha^2\right) + \left(15\alpha^6 - 24\alpha^4 + 9\alpha^2\right)\cos(2\theta) + 6\left(\alpha^6 - \alpha^4\right)\cos(4\theta) + \alpha^6\cos(6\theta)$

$\mathrm{T}_7(\alpha\cos\theta) = 7\left(5\alpha^7 - 10\alpha^5 + 6\alpha^3 - \alpha\right)\cos(\theta) + 7\left(3\alpha^7 - 5\alpha^5 + 2\alpha^3\right)\cos(3\theta) + 7\left(\alpha^7 - \alpha^5\right)\cos(5\theta) + \alpha^7\cos(7\theta)$

Series Expansion: The Tylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

Trigonometry: Useful Identities

- $A_1\cos\theta + A_2\sin\theta = \sqrt{A_1^2 + A_2^2}\cos(\theta - \theta_0),\ \tan\theta_0 = \frac{A_2}{A_1}$
- $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$
- $\sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$

Trigonometry: Powers of Cosine

$\frac{1}{2}(2\cos\theta)^2 = 1 + \cos(2\theta)$

$\frac{1}{2}(2\cos\theta)^3 = 3\cos(\theta) + \cos(3\theta)$

$\frac{1}{2}(2\cos\theta)^4 = 3 + 4\cos(2\theta) + \cos(4\theta)$

$\frac{1}{2}(2\cos\theta)^5 = 10\cos(\theta) + 5\cos(3\theta) + \cos(5\theta)$

$\frac{1}{2}(2\cos\theta)^6 = 10 + 15\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta)$

$\frac{1}{2}(2\cos\theta)^7 = 35\cos(\theta) + 21\cos(3\theta) + 7\cos(5\theta) + \cos(7\theta)$

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